Equations governing 1D hydraulics:

1. **Mass conservation:**
   \[
   \frac{\partial A}{\partial t} + \frac{\partial}{\partial x} (Au) = 0
   \]

2. **Momentum budget:**
   \[
   \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -g \frac{\partial h}{\partial x} + gS - C_D \frac{u^2}{R}
   \]

where

- \( h \) = water depth, \( A(h) \) = cross-sectional area, \( u \) = velocity
- \( S \) = slope, \( R \) = hydraulic radius = area/wetted perimeter
Restrict attention to steady states.

Equations reduce to:

1. **Mass conservation**
   \[
   \frac{d}{dx}(Au) = 0 \quad \Rightarrow \quad Au = Q = \text{constant}
   \]

2. **Momentum budget**
   \[
   u \frac{du}{dx} + g \frac{dh}{dx} = gS - C_D \frac{u^2}{R}
   \]

Two categories of steady flows:

1. **Rapidly varied flow**
   when channel slope changes quite abruptly
   no time for friction → assume no friction

2. **Gradually varied flow**
   when changes in slope occur over long distances
   friction has time to act → friction must be retained
Steady, gradually varied flows

Friction is retained \( \rightarrow \) the \( C_d \) term is kept.

Steadiness yields: \( u = \frac{Q}{A(h)} \)

\[
u \frac{du}{dx} + g \frac{dh}{dx} = gS - C_d \frac{u^2}{R} \quad \rightarrow \quad -\frac{Q^2}{A^3} \frac{dA}{dx} + g \frac{dh}{dx} = gS - C_d \frac{Q^2}{RA^2}
\]

or

\[
\left(1 - \frac{Q^2}{gA^2} h\right) \frac{dh}{dx} = S - \frac{C_d Q^2}{gRA^2} \quad \text{in which} \quad \bar{h} = \text{average depth} = \frac{A}{W}, \quad W = \frac{dA}{dh}
\]

or

\[
\left(1 - Fr^2\right) \frac{dh}{dx} = S \left(1 - \frac{C_d Q^2}{gSRA^2}\right)
\]

Wide rectangular channel, so that \( R = \frac{Wh}{W + 2h} \approx h \)

\[
\left(1 - \frac{h^3}{h^3}\right) \frac{dh}{dx} = S \left(1 - \frac{h^3}{h^3}\right)
\]

in which

\[
h_c = \left(\frac{Q^2}{gW^2}\right)^\frac{1}{3} = \text{critical depth (depth if} \ Fr = 1) \quad Fr > 1 \text{ for} \ h < h_c
\]

\[
h_n = \left(\frac{C_d Q^2}{S gW^2}\right)^\frac{1}{3} = \text{normal depth (depth if uniform flow)}
\]
Two possibilities:

1. $0 < h < h_c$ (supercritical flow)
   \[ \frac{dh}{dx} > 0 \quad \text{(case A1)} \]

2. $h_c < h$ (subcritical flow)
   \[ \frac{dh}{dx} < 0 \quad \text{(case A2)} \]

Three possibilities:

1. $0 < h < h_c$ \[ \frac{dh}{dx} > 0 \quad \text{(case M1)} \]

2. $h_c < h < h_n$ \[ \frac{dh}{dx} < 0 \quad \text{(case M2)} \]

3. $h_n < h$ \[ \frac{dh}{dx} > 0 \quad \text{(case M3)} \]
Case of a steep slope:

$S > 0$ but large

$h_n > 0$ and small,
so that

$h_n < h_c$

\[
\left(1 - \frac{h_n^3}{h^3}\right) \frac{dh}{dx} = S \left(1 - \frac{h_c^3}{h^3}\right)
\]

3 possibilities again:

1. $0 < h < h_n \Rightarrow \frac{dh}{dx} > 0$ (case S1)
2. $h_n < h < h_c \Rightarrow \frac{dh}{dx} < 0$ (case S2)
3. $h_c < h \Rightarrow \frac{dh}{dx} > 0$ (case S3)

Combinations:
More combinations:

Lake discharge problem:

The question is:
Given the water level in a lake and the elevation of the bottom at the edge, what is the discharge?
Answer: It very much depends whether the downstream channel has a **mild** or **steep** slope.

Mild slope if $S < C_D$
Steep slope if $S > C_D$

Since $C_D$ is a function of the water depth $h$, we do not know a priori whether the slope is mild or steep, but we start with a reasonable guess.

---

**Case of a mild slope:**

$$S < C_D$$

Of the three possible solutions (M1, M2 and M3), both M2 and M3 work. And, the choice depends on what happens far downstream. Assume that the channel is long so that it does not matter which case we choose.

At the lake sill, the water depth is $h_s = \left( \frac{C_D Q^2}{g SW^2} \right)^{\frac{1}{3}}$ then check on $C_D$.

Bernoulli principle from inside lake to sill point:

$$\frac{0}{2} + gH = \frac{u^2}{2} + gh_n \rightarrow u = \sqrt{2g(H-h_n)}$$

$$Q = Wh_u = \left[ \frac{1}{2} \left( \frac{S}{C_D} \right)^{\frac{2}{3}} + \left( \frac{C_D}{S} \right)^{\frac{1}{3}} \right]^{\frac{1}{2}} WH \sqrt{gH}$$
Case of a steep slope:

\[ S > C_D \]

Of the three possible solutions (S1, S2 and S3), only S2 works.

Flow down the exit channel is supercritical (fast and thin), while the flow was obviously subcritical inside the lake (thick and slow). The flow must therefore be critical at the sill.

Bernoulli principle from inside lake to sill point:

\[
\frac{0}{2} + gH = \frac{u_c^2}{2} + gh_c = \frac{gh_l}{2} + gh_c \quad \rightarrow \quad h_c = \frac{2}{3} H \quad (\text{then check on } C_D)
\]

\[
u_c = \sqrt{gh_c} = \sqrt[3]{\frac{2}{3} gH}
\]

\[
Q = Whu = Whu_c = \left(\frac{2}{3}\right)^{\frac{3}{2}} WH \sqrt{gH} = 0.544 \quad WH \sqrt{gH}
\]