ENVIRONMENTAL FLUID MECHANICS

Partial Differential Equations
governing the motion of environmental fluids

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Procedure

The theory proceeds with the consideration of an infinitesimal control volume, of dimensions $\Delta x$, $\Delta y$ and $\Delta z$ along the $x$, $y$ and $z$-axes, respectively, and over which mass, momentum, energy and other budgets are performed.

Once budgets are established, the control volume is shrunk to point by taking the limit:

$\Delta x \to 0$, $\Delta y \to 0$ and $\Delta z \to 0$

The volume enclosed is $V = \Delta x \cdot \Delta y \cdot \Delta z$. 
Budget for an arbitrary quantity $c$

Accumulation $= \sum$ imports $- \sum$ exports

$+ \sum$ sources $- \sum$ sinks

$V \frac{dc}{dt} = \sum c_i A_i + S$

$$\Delta x \Delta y \Delta z \frac{dc}{dt} = + c \left( x - \frac{\Delta x}{2}, y, z \right) u \left( x - \frac{\Delta x}{2}, y, z \right) \frac{\Delta x \Delta y \Delta z}{\Delta x} \text{ entering from left}$$

$$- c \left( x + \frac{\Delta x}{2}, y, z \right) u \left( x + \frac{\Delta x}{2}, y, z \right) \Delta y \Delta z \text{ exiting from right}$$

$$+ c \left( x, y + \frac{\Delta y}{2}, z \right) v \left( x, y + \frac{\Delta y}{2}, z \right) \Delta x \Delta z \text{ entering from front}$$

$$- c \left( x, y + \frac{\Delta y}{2}, z \right) v \left( x, y + \frac{\Delta y}{2}, z \right) \Delta x \Delta y \text{ exiting from back}$$

$$+ c \left( x, y + \frac{\Delta y}{2}, z \right) \left( x, y + \frac{\Delta y}{2}, z \right) \Delta x \Delta y \text{ entering from bottom}$$

$$- c \left( x, y + \frac{\Delta y}{2}, z \right) \left( x, y + \frac{\Delta y}{2}, z \right) \Delta x \Delta y \text{ exiting from top}$$

$+ S \text{ sum of internal sources and sinks}$

Divide the preceding equation by $\Delta x \Delta y \Delta z$ and move the imports and exports to the left-hand side:

$$\frac{dc}{dt} = \frac{(cu)_{x=\Delta x/2} - (cu)_{x=0}}{\Delta x}$$

$$+ \frac{(cv)_{y=\Delta y/2} - (cv)_{y=0}}{\Delta y}$$

$$+ \frac{(cw)_{z=\Delta z/2} - (cw)_{z=0}}{\Delta z} \frac{S}{\Delta x \Delta y \Delta z}$$

Now shrink the volume to a point by taking the limit $\Delta x \to 0$, $\Delta y \to 0$ and $\Delta z \to 0$:

$$\frac{\partial c}{\partial t} + \frac{\partial (cu)}{\partial x} + \frac{\partial (cv)}{\partial y} + \frac{\partial (cw)}{\partial z} = s$$

in which $s$ is the net source-minus-sink of the quantity per unit volume ($s = S/\Delta x \Delta y \Delta z$).
Conservation of mass

The quantity \( c \) is then mass/volume = density, noted \( \rho \).

Mass conservation implies that there is no source or sink \( (s = 0) \).

The budget equation then becomes:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0
\]

After expanding the derivatives of products, the equation can also be spelled out as:

\[
\rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0
\]

all small terms because variations in density are much smaller than the density itself in environmental fluids.

Conservation of momentum

For momentum, sources are forward (accelerating) forces and sinks are backward (braking) forces.

Let us begin with the pressure forces on the small control volume.

Pressure = force per area
\( \rightarrow \) force = pressure x area

In the \( x \)-direction, there are two pressure forces:

\[
p \left( x + \frac{\Delta x}{2}, y, z \right) \Delta y \Delta z \text{ in the positive } x \text{-direction}
\]

\[
p \left( x - \frac{\Delta x}{2}, y, z \right) \Delta y \Delta z \text{ in the negative } x \text{-direction}
\]

The net force in the \( x \)-direction is the difference of the two, which can be recast as (using a Taylor expansion):

\[
\frac{\partial p}{\partial x} \Delta x \Delta y \Delta z + O(\Delta x^2 \Delta y \Delta z)
\]

Similarly in the \( y \)- and \( z \)-directions, the net forces are, respectively:

\[
\frac{\partial p}{\partial y} \Delta x \Delta y \Delta z + O(\Delta x \Delta y^2 \Delta z)
\]

\[
\frac{\partial p}{\partial z} \Delta x \Delta y \Delta z + O(\Delta x \Delta y \Delta z^2)
\]

Note that it is not the pressure that moves the fluid, but the pressure gradient, that is, the pressure difference between one side and the other.
Next in consideration is the gravitational force, which acts purely in the negative \( z \)-direction (vertical).

\[
\text{Force of gravity} = \text{mass} \times \text{gravitational acceleration} = mg
\]

\[
\text{Density} = \text{mass per volume} \rightarrow \text{mass} = \text{density} \times \text{volume}
\]

\[
m = \rho V = \rho \Delta x \Delta y \Delta z
\]

The gravitational force is then: 0 in the \( x \)- and \( y \)-directions

\[- \rho g \Delta x \Delta y \Delta z \text{ in the } z \text{-direction (vertical)}\]

Let us now consider the stresses that act on the small control volume.

Unlike pressure, which always acts perpendicularly to a face, stresses may be oblique and are thus to be considered as 3D vectors, with three components on each face.

That makes 9 components altogether.

In other words, stresses form a vector of vectors. This is a tensor:

\[
\begin{pmatrix}
\tau_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \tau_{yy} & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \tau_{zz}
\end{pmatrix}
\]

In the \( x \)-direction alone, there are 6 forces to reckon with:

\( \tau_x \) on left and right sides,
\( \tau_y \) on front and back side, and
\( \tau_z \) on bottom and top sides.

The net force in the \( x \)-direction can be expressed as (in similarity with pressure force):

\[
\left[ \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right] \Delta x \Delta y \Delta z + O(\Delta x^2 \Delta y \Delta z, \Delta x \Delta y^2 \Delta z, \Delta x \Delta y \Delta z^2)
\]

with similar expressions for the \( y \)- and \( z \)-directions.
Putting all the pieces together, we obtain:

\[
\begin{align*}
\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} &= \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \\
\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} &= \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \\
\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} &= \frac{\partial \tau_{zz}}{\partial z} + \rho g,
\end{align*}
\]

inertia

also called advection

pressure

force

friction

force

gravity

force

Some simplifications can be performed via "scale analysis".

Scale Analysis

For every variable in the mathematical formulation, a "scale" is introduced, which measures the "size" or "order of magnitude" of the variable.

The following scales are used:

- \(L\) for horizontal distances \(x\) and \(y\)
- \(H\) for the vertical distance \(z\) \((H << L\) in environmental fluid systems\)
- \(T\) for time \(t\) \(\) (not to be confused here with temperature!\)
- \(U\) for the horizontal velocity components \(u\) and \(v\)
- \(W\) for the vertical velocity component \(w\) \((W << U\) as a consequence of \(H \ll L\)\)
- \(P\) for the pressure
- \(\Delta P\) for the dynamic pressure
- \(\Delta \rho\) for density anomalies \((\Delta \rho \ll \rho_0\) in environmental fluids\)

A derivative "scales" like the ratio of its variables:

\[
\frac{\partial u}{\partial x}, \frac{U}{T}, \frac{\partial u}{\partial x}, \frac{U}{L}, \frac{\partial w}{\partial z}, \frac{W}{H}, \text{ etc.}
\]
Hurricane Frances during her passage over Florida on 5 September 2004. The diameter of the storm is about 830 km and its top wind speed approaches 200 km/hour (= 56 m/s). Its track displayed appreciable changes in speed and direction over 2-day intervals.

From this, we can use the following scales:

\( L = 800 \text{ km} \), \( U = 60 \text{ m/s} \), \( T = 2 \text{ days} \)

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**Froude number**

From the hydrostatic balance, we can relate the pressure scale to other variables:

\[
\frac{\partial p}{\partial z} = -\rho g \rightarrow \frac{P}{H} - \rho g \rightarrow P - \rho g H
\]

If this pressure is the operating pressure in the horizontal momentum equations (not always the case!), then

\[
\frac{1}{\rho H} \frac{\partial p}{\partial x} \text{ and } \frac{1}{\rho H} \frac{\partial p}{\partial y} - \frac{P}{\rho L} - \frac{\rho g H}{\rho L} = \frac{g H}{L}
\]

In the horizontal momentum equations, the inertial terms scale as:

\[
u \frac{\partial u}{\partial x} \text{ and other similar terms} - U \frac{U}{L} = \frac{U^2}{L}
\]

The ratio of inertia to pressure force is

\[
\frac{\text{inertia}}{\text{pressure force}} = \frac{U^2 / L}{g H / L} = \frac{U^2}{g H}
\]

The Froude number is defined as

\[
Fr = \frac{U}{\sqrt{g H}}
\]

- \( Fr < 1 \): Flow is fluvial
- \( Fr = 1 \): Flow is critical
- \( Fr > 1 \): Flow is torrential
Transition from subcritical to supercritical flow

Sugar River passing through Croydon, New Hampshire (USA)
(Photo by Benoit Cushman-Roisin)

Richardson number

Most often, however, it is not the main hydrostatic part of the pressure that drives the motion but the hydrostatic pressure anomaly caused by thermally-induced changes in density.

The relevant pressure scale for the estimation of horizontal pressure forces is then:

$$\Delta P = gH\Delta \rho$$

And the horizontal pressure forces scale as

$$\frac{1}{\rho_0} \frac{\partial p}{\partial x} \quad \text{and} \quad \frac{1}{\rho_0} \frac{\partial p}{\partial y} = \frac{\Delta P}{\rho_0 L} = \frac{gH \Delta \rho}{\rho_0 L}$$

The comparison with inertia yields

$$\frac{\text{inertia}}{\text{pressure force}} = \frac{U^2/L}{gH\rho_0/L} = \frac{\rho_0 U^2}{gH\rho_0}$$

The Richardson number is defined as:

$$Ri = \frac{gH\rho_0}{\rho_0 U^2}$$

- $Ri < 1$: Stratification is weak and vulnerable to mixing
- $Ri \approx 1$: Stratification is important and may be affected by mixing
- $Ri >> 1$: Stratification is dominant and resists mixing
Vorticity

Mathematical definition:
Vorticity is the curl of the velocity vector
\[ \vec{\omega} = \nabla \times \vec{u} \]

Vorticity is thus a 3D vector of which the components are
\[ \omega_x = \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}, \quad \omega_y = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}, \quad \omega_z = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \]

Physical meaning:
Vorticity measures the amount of spinning in the flow.

Example: Solid-body rotation
\[ u = -\Omega y, \quad v = +\Omega x, \quad w = 0 \rightarrow \vec{\omega} = (0,0,2\Omega) \]

Why is vorticity important in fluid mechanics?
It is essentially because the pressure force, being a normal force, exerts no torque on the fluid and is thus incapable of adding or subtracting “spin” to the flow.

Mathematically, this is due to the fact that the curl of a gradient is always zero:
\[ \nabla \times \nabla f = 0 \]

As a result, vorticity is a flow property that enjoys a higher degree of conservation than momentum.

To see this, we take the curl \( \nabla \times \) of the momentum equations. The outcome in the absence of frictional forces is:
\[ \frac{\partial \vec{\omega}}{\partial t} + (\vec{u} \cdot \nabla) \vec{\omega} - (\vec{\omega} \cdot \nabla) \vec{u} = -\rho_0 \vec{b} \]

In which the vector \( \vec{b} \) has the following components:
\[ b_x = -\frac{\rho g}{\rho_s} \frac{\partial \rho}{\partial x}, \quad b_y = \frac{\rho g}{\rho_s} \frac{\partial \rho}{\partial y}, \quad b_z = 0 \]
Vorticity dynamics in a river bend

Friction against the channel bed causes a vertical shear in the flow, $u(z)$, to which corresponds a transverse vorticity component $\omega_y$ directed to the left of the flow.

The preservation of the $\omega_y$ vorticity component past the 90° bend corresponds to a secondary circulation in which the flow near the surface is drifting toward the outer shore and the flow near the bottom is drifting toward the inner shore.

This phenomenon may also be explained without recourse to vorticity dynamics by considering the centrifugal force involved in the bend, which causes the water level to rise a little across the stream in the bend, from inner shore to outer shore. The corresponding transverse pressure gradient, which is the same at all depths by virtue of hydrostaticity, is insufficient to balance the greater centrifugal force near the surface and at the same time excessive in balancing the weaker centrifugal force near the bottom. Hence, the surface water drifts outward due to excess centrifugal force and the bottom water drifts inner due to excess pressure force.

90° bend in Mascoma River, Lebanon, New Hampshire (USA)

The near vertical shore on the outside of the curve (left side of photo) is in sharp contrast to the relatively flat shore on the inside of the curve (right side of photo).

This is because the fast outward on the surface causes erosion of the outer shore while the slow inward flow along the bottom causes sedimentation near the inner shore.
A serpentine flow pattern in the disinfection step of wastewater treatment not only makes the geometry more compact but it also creates a secondary circulation at every bend that enhances the mixing of the disinfection agent (usually a chlorinated compound) into the water being treated.
Bjerknes’ Circulation Theorem

Bjerknes’ circulation theorem is Kelvin’s circulation theorem extended to flows with density variations:

Bjerknes’ circulation is easy to understand physically:

Suppose that in addition to the typical vertical density gradient \( \frac{d\rho}{dz} < 0 \) (lighter fluid floating on top of denser fluid), there also exists a horizontal density gradient \( \frac{d\rho}{dx} < 0 \).

Because gravity pushes the denser fluid to intrude under the lighter fluid, a horizontal motion is induced as depicted in the sketch on the right, which has the net effect of bringing all the lighter fluid to the top and all the denser fluid to the bottom.

\[
\frac{d\Gamma}{dt} = -\frac{g}{\rho_0} \int \rho \, dz > 0
\]

This loop flow possesses a circulation in the direction indicated in the figure, which corresponds to a vorticity aligned with the \(-y\) direction.

A manifestation of circulation dynamics – The sea breeze

Given the equation of state

\[
\rho = \rho_0 [1 - \alpha (T - T_0)]
\]

the preceding equation can be rewritten as and approximated to:

\[
\frac{dT}{dt} = \frac{\alpha g}{H} \int [T(A \rightarrow B) - T(D \rightarrow C)] \, dz
\]

\[
= \alpha g H (T_{sl} - T_{sc})
\]