Example of stirring by turbulent flow

Pierre Welander (1955)
Consider the chaotic trajectory $\vec{x}(t)$ of a point in a turbulent flow, starting at the origin (like a smoke particle coming out of a chimney) and define the time-dependent quantity

$$R^2(t) = \langle \vec{x}(t) \cdot \vec{x}(t) \rangle$$

to measure its (squared) distance from the origin by time $t$. Here, the symbol $\langle \rangle$ indicates an average over the turbulent fluctuations.

The rate of change of this quantity over time is:

$$\frac{dR^2}{dt} = 2 \langle \vec{x}(t) \cdot \vec{u}(t) \rangle = 2 \int_{t} \vec{u}(t')d't\vec{u}(t) \rangle \quad \text{since } \vec{u}(t) = \frac{d\vec{x}}{dt} = \text{velocity.}$$

Switching the averaging operator with the time integration, we can rewrite this as:

$$\frac{dR^2}{dt} = 2 \int_{t} \langle \vec{u}(t') \cdot \vec{u}(t) \rangle d't = 2 \int_{t} \langle \vec{u}(t-t) \cdot \vec{u}(t) \rangle d\tau$$

in which $\tau$ is a time delay ($0 < \tau < t$).

(G. I. Taylor, 1921)
**Stratification**

Entrance to Hanover, New Hampshire
Morning of 13 October 2010
Fog layer after a cold night

Mixing of a two-layer stratification – Can it be spontaneous?

If complete mixing occurs, the final state must be characterized by:

\[ U = \frac{H_1 U_1 + H_2 U_2}{H_1 + H_2} \]
\[ \rho = \frac{H_1 \rho_1 + H_2 \rho_2}{H_1 + H_2} \]
Change in Kinetic Energy:

\[
\begin{align*}
KE_{\text{initial}} &= \int_0^L \frac{1}{2} \rho_0 \mu^2 dz = \frac{1}{2} \rho_0 u_1^2 H_1 + \frac{1}{2} \rho_0 u_2^2 H_2 \\
KE_{\text{final}} &= \frac{1}{2} \rho_0 U^2 H = \frac{1}{2} \rho_0 \left( \frac{H_1 U_1 + H_2 U_2}{H_1 + H_2} \right)^2
\end{align*}
\]

\[
KE_{\text{drop}} = KE_{\text{initial}} - KE_{\text{final}} = \frac{1}{2} \rho_0 \frac{H_1 H_2 U_1^2 + H_2 H_1 U_2^2 - 2 H_1 H_2 U_1 U_2}{H_1 + H_2}
\]

Change in Potential Energy:

\[
\begin{align*}
PE_{\text{initial}} &= \int_0^L \rho(z) g(z) dz = \int_{H_1}^{H_2} \rho \left( g H_1 + \frac{g}{2} \right) dz \\
&= \frac{1}{2} \rho_0 g H_1^2 + \frac{1}{2} \rho_0 g (2H_1 H_2 + H_2^2) \\
PE_{\text{final}} &= \int_0^L \rho g(z) dz = \rho g \frac{H_2^2}{2} = \frac{g}{2} (\rho_1 H_1 + \rho_2 H_2)(H_1 + H_2)
\end{align*}
\]

\[
PE_{\text{gain}} = PE_{\text{final}} - PE_{\text{initial}} = -\frac{1}{2} (\rho_2 - \rho_1) g H_1 H_2
\]

Is there enough Kinetic Energy release to provide for the necessary Potential Energy increase?

The answer is yes as long as

\[
\frac{1}{2} \rho_0 \frac{H_1 H_2 (U_1 - U_2)^2}{H_1 + H_2} > \frac{1}{2} (\rho_2 - \rho_1) g H_1 H_2 \quad \Rightarrow \quad \rho_0 (U_1 - U_2)^2 > gH(\rho_2 - \rho_1)
\]