LECTURE 6

Combinatorics and Occupancy Problems

Reading: Sections 2.2-2.4

When all outcomes of an experiment are equally likely, the probability of a single outcome is

$$p(\omega_i) = \frac{1}{\text{Total number of outcomes}} = \frac{1}{|\mathcal{S}|}$$

and the probability of an event E is

$$P(E) = \sum_{\omega_i \in E} p(\omega_i) = \text{Number of successful outcomes} \times \frac{1}{|\mathcal{S}|} = \frac{|E|}{|S|}\ .$$

For all but the simplest sample spaces, determining the size of E by listing all the outcomes in the event is a tedious process. Fortunately, there are easier ways to figure out how many objects are in a set. This is the mathematical subject called *combinatorial analysis* or *combinatorics.* [1]

6.1. The sum rule

The size of two disjoint sets is the sum of their sizes: $|A \cup B| = |A| + |B|$.

Example — Flip a coin. If it comes up heads, roll a die. If it comes up tails, flip it again. How many outcomes are possible? There are six outcomes associated with getting a head on the coin toss. There are two outcomes associated with getting a tail. The two sets of outcomes are disjoint, so the total number of outcomes is $6 + 2 = 8$.

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[1] For further reading see K.P. Bogart, *Discrete Mathematics* (Heath, 1988), Chapter 6, and Feller, Volume 1, Chapter 2.

6.2. The product rule

The size of the cartesian product of two sets is the product of their sizes: $|A \times B| = |A| \cdot |B|$.

If there are m ways to do A and n ways to do B, then there are mn ways to do A followed by B:

$$\{a_1, a_2, \ldots, a_m\} \times \{b_1, b_2, \ldots, b_n\} = \left\{ \begin{matrix} (a_1, b_1) & (a_1, b_2) & \cdots & (a_1, b_n) \\ (a_2, b_1) & (a_2, b_2) & \cdots & (a_2, b_n) \\ \vdots & \vdots & \ddots & \vdots \\ (a_m, b_1) & (a_m, b_2) & \cdots & (a_m, b_n) \end{matrix} \right\}$$

Examples

- Sampling with replacement — There are 2 alternatives for a coin toss. There are $2 \times 2 = 4$ possible outcomes of tossing two coins. There are $2 \times 2 \times \cdots \times 2 = 2^N$ ways to flip N coins. There are 2^N ways to assign 1's and 0's in an N-bit binary number. There are 6×6 ways to roll two dice.

- Sampling without replacement — A poker hand contains five cards. There are 52 alternatives for the first card, 51 for the second, 50 for the third, 49 for the fourth, and 48 for the fifth. The total number of ways to be dealt a hand is $52 \times 51 \times 50 \times 49 \times 48 = \frac{52!}{47!}$ (a very large number). The actual number of poker hands is smaller than this (see below).

- Set A contains n objects, and set B contains m objects. Without loss of generality, let $n \le m$. Each object in A is to be paired with a unique object from B. A set of pairings $\{(a_1, b_{i_1}), (a_2, b_{i_2}), \ldots, (a_n, b_{i_n})\}$ is called a configuration. How many possible configurations are there?

6.3. Permutations

An *ordered arrangement* of the items in a set is called a *permutation* of the set. An ordered arrangement of r items selected from a set of n items is called an *r-element permutation* of the set.

The number of r-element permutations of an n-element set, denoted P_n^r, is

$$\begin{aligned} P_n^r &= n \cdot (n-1) \cdots (n-r+1) = \frac{n \cdot (n-1) \cdots (n-r+1) \cdot (n-r) \cdot (n-r-1) \cdots 1}{(n-r) \cdot (n-r-1) \cdots 1} \\ &= \frac{n!}{(n-r)!} \end{aligned} \tag{6.1}$$

Examples

- The permutations of the list *abc* are *abc*, *acb*, *bac*, *bca*, *cab*, *cba*. The total number of permutations is 6: 3 ways to pick the first character, 2 ways to pick the second character, and 1 way to pick the third character. Note $\frac{3!}{(3-3)!} = \frac{3!}{1} = 3! = 6$.

- Revisit the example above concerning the pairing of n objects from set A with m objects from set B, with $n \leq m$. Line up the elements of A in a fixed order. Make a configuration by selecting an n-element permutation of B, $\{b_{i_1}, b_{i_2}, \ldots, b_{i_n}\}$, and pairing the a's with the b's. The number of configurations is the number of permutations, P_n^m.

- The birthday problem. In a class of N students, what is the probability that at least two people have the same birthday? (*Hint:* It is easier to find the probability that all N birthdays are different.) How large does the class have to be in order for the probability to be 0.5?

6.4. Combinations

An *unordered arrangement* of r items selected from an n-element set, *i.e.*, an r-element subset, is called a *combination.*

Each of the r-element combinations (unordered arrangements) can be ordered in $r!$ ways. The total number of ordered arrangments (permutations) is $r!$ times the number of unordered arrangements (combinations). The number of r-element combinations of n objects is denoted C_r^n.

$$r! \times \text{Number of unordered arrangements} = \text{Number of ordered arrangements}$$

$$r! \times C_r^n = P_r^n = \frac{n!}{(n-r)!}$$

$$\therefore \quad C_r^n = \frac{n!}{r!(n-r)!} \tag{6.2}$$

Another way of writing C_r^n is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \tag{6.3}$$

and is sometimes read "n choose r".

Examples

- The three-element permutations of the set $\{a, b, c, d\}$ are:

  $$\begin{array}{cccc} abc & abd & acd & bcd \\ acb & adb & adc & bdc \\ cab & dab & dac & dbc \\ cba & dba & dca & dcb \\ bca & bda & cda & cdb \\ bac & bad & cad & cbd \end{array}$$

  The number of permutations is $\frac{4!}{(4-3)!} = \frac{4!}{1!} = 4! = 24$. On the other hand, the three-element combinations of $\{a, b, c, d\}$ are just:

  $$\begin{array}{cccc} abc & abd & acd & bcd \end{array}$$

  The number of combinations is considerably smaller, only 4. Each column of the array of permutations consists of rearrangements of one of these four combinations.

- There are $\frac{52!}{47!}$ ways to be dealt a poker hand, but the ordering of the cards in a hand is irrelevant to the game. There are actually $C^{52}_5 = \binom{52}{5}$ different poker hands after the rearrangements are removed. How many ways are there to get a pair? What is the probability of being dealt a pair?

- In Engs 31, a group of students built a version of the Minesweeper game. Their game board was 4×4 squares. At the beginning of a game, four mines had to be "randomly" placed on the board. Here's how they did it. There were $C^{16}_4 = \binom{16}{4} = 1820$ different ways to put four mines on the board. Each square needed 1 bit to store the presence or absence of a mine, so to store a whole game board required 16 bits = 2 bytes. The students found they could store all the possible game boards in $\binom{16}{4} \times 2 = 3640$ bytes, which are comfortably accomodated by a $4K \times 8$ read-only memory.

- When I first came to Dartmouth, the D-plan was more complicated than it is now. Students were required to spend the freshman year on-campus, at least one summer term on campus, and a total of eleven (not twelve) terms at the college to graduate. How many different D-plans were there, and how does that number compare to the number of D-plans today?

- Widget inspection. Suppose a batch of N widgets has M defective ones. If r widgets are randomly sampled from the batch, what is the probability that any of them are defective? What is the probability that exactly k of them are defective?

6.5. Binomial coefficients

The symbol $\binom{n}{r}$ is also called a *binomial coefficient*, because

$$(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r \ . \tag{6.4}$$

For example,

$$\begin{aligned}(x+y)^3 &= \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3 \\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

Why is this so? Consider $(x+y)^3$ as an example.

$$\begin{aligned}(x+y)^3 = (x+y)(x+y)(x+y) &= (x+y)(xx+xy+yx+yy) \\ &= xxx+xxy+xyx+xyy+yxx+yxy+yyx+yyy.\end{aligned}$$

Each term may be regarded as selecting an x or a y from each of the three $(x+y)$ factors. There is only one way to select 3 x's: "3 choose 3" = $\binom{3}{3} = 1$. To select 2 x's from the three factors, there are $\binom{3}{2} = 3$ ways. To select 1 x, there are $\binom{3}{1} = 3$ ways. Finally, to select 0 x's, there is $\binom{3}{0} = 1$ way. In general, the number of ways to select $n-r$ x's and r y's is $\binom{n}{r}$.

If we set $x = y = 1$ in (6.4), we get

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n \tag{6.5}$$

Examples

- How many binary numbers can be represented by n bits? Approach this by counting the number of ways to set r of the n bits to one. There are $\binom{n}{0}$ with 0 ones, $\binom{n}{1}$ with 1 one, etc. Adding these up, we have

  $$\#n\text{-bit numbers} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

- The *power set* $\mathcal{P}(\mathcal{S})$ of a set $\mathcal{S}$ is the set of all subsets of $\mathcal{S}$. How many are there? If $|\mathcal{S}| = n$, then the number of r-element subsets of $\mathcal{S}$ is $\binom{n}{r}$. Adding these up for $r = 0$ to n gives, according to (6.5), $|\mathcal{P}(\mathcal{S})| = 2^n = 2^{|S|}$. Another way to approach this is to note that every possible subset of $\mathcal{S}$ can be encoded by an n-bit binary number $b_1b_2\cdots b_n$, where $b_k = 1$ means that the k^{th} element of $\mathcal{S}$ is in the subset, and $b_k = 0$ means that it isn't. The number of n-bit binary numbers is, as shown above, equal to 2^n.

Two other facts about the binomial coefficients are

$$\binom{n}{r} = \binom{n}{n-r} \tag{6.6}$$

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \tag{6.7}$$

6.6. Occupancy problems

Recall the definitions of permutation and combination. If a set has n elements, a combination is a subset — the order of the elements in a subset does not matter. The number of r-element subsets is

$$C_r^n, C(n,r), nCr = \binom{n}{r} = \frac{n!}{r!(n-r)!} .$$

A reordering of a list of objects is a permutation. A list of length r has $r!$ permutations (r ways to choose the first element, $r-1$ ways to choose the second, etc). The number of r-element permutations of the elements in a set of size n is

$$P_r^n, P(n,r), (n)_r, nPr = r!\binom{n}{r} = \frac{n!}{(n-r)!} .$$

Many applications can be reduced to the problem of assigning objects to categories, *e.g.*, placing balls into boxes. These are called *occupancy problems.* The solutions break down according to whether the balls are distinguishable (*i.e.*, must be labelled) or indistinguishable. If the balls are indistinguishable, then their assignment amounts to a generalization of selecting subsets. If they are distinguishable, then they can be permuted before being assigned to their boxes. We can expect that there will be more possible assignments for distinguishable balls than for indistinguishable ones.

Here are some typical occupancy problems.

1. *How many ways can r identical, distinguishable objects be assigned to n cells, with no limit on the population of a cell?* There are n possible locations for each object, so

$$\#\text{ possible assignments} = n \cdot n \cdots n = n^r . \tag{6.8}$$

   In physics, this leads to a particular model called *Maxwell-Boltzmann* statistics.

2. *How many ways can r identical, indistinguishable objects be assigned to n cells, with no limit on the population of a cell?* Suppose you line all the objects up in a row. Because they are indistinguishable, the order of the objects doesn't matter. The process of assigning them to n boxes is equivalent to placing $n+1$ walls to separate them into groups, *e.g.*, for 9 balls and 6 boxes, the following is a possible assignment.

$$|****|*|-|**|-|*|$$

   The $-$ denotes an empty box. Different assignments are made by moving the walls around. How many ways can the walls be placed? The walls on the ends are fixed, leaving $n-1$ which can be moved. The number of balls + movable walls is $n+r-1$, and this is also the number of places where a wall can be placed. The number of arrangements is the number of ways that the $n-1$ walls can be assigned to $n+r-1$ slots, which is

$$\#\text{ arrangements} = C_{n-1}^{n+r-1} = \binom{n+r-1}{n-1} = \binom{n+r-1}{r} . \tag{6.9}$$

   Another way: There are $n+r-1$ slots, so there are $(n+r-1)!$ ways to arrange the balls and walls, if they are distinguishable. However, they aren't, so we divide by $(n-1)!$ to remove the permutations of the walls, and then by $r!$ to remove the permutations of the balls. In physics, this leads to a model called *Bose-Einstein* statistics, and particles which obey this model are called *bosons*.

3. *How many ways can r identical, indistinguishable objects be assigned to n cells, with the restriction that a cell can hold no more than one object?* Obviously, for this to be possible we must have $r \leq n$. Assume this is so. There will be r cells holding one object, and $n-r$ which are empty. The "balls and walls" method used above can be adapted here, if, instead of using "$|$" and "$*$", we use "$|*$" and "$|$". There are r "$|*$"s and $n-r$ "$|$"s, for a total of n slots where things can be placed. Then, using the previous result, the answer is
$$\#\text{ arrangements} = C^n_{n-r} = \binom{n}{n-r} = \binom{n}{r}\,. \tag{6.10}$$
In physics, this leads to a model called *Fermi-Dirac* statistics, and particles which obey this model are called *fermions*. The constraint is a consequence of the Pauli exclusion principle.

4. *How many ways can r identical, indistinguishable objects be placed into n cells, so that r_1 go into the first cell, r_2 go into the second cell, etc?* The r_k are called *occupancy numbers*, and we are asking how many ways we can achieve the particular occupancy vector $(r_1, r_2, \ldots, r_n)$. For the first cell, we are asking how many r_1-length subsets of r objects there are. This is simply $\binom{r}{r_1}$. For the second cell, there are $r - r_1$ objects left, and we are selecting a subset of size r_2. There are $\binom{r-r_1}{r_2}$ ways to do this. Continue to the last cell, where r_n objects are selected from the $r - r_1 - r_2 - \cdots r_{n-1} = r_n$ remaining objects. There is only one way to do this. By the product rule, we multiply these numbers together, and obtain
$$\begin{aligned}\#\text{ arrangements} &= \binom{r}{r_1}\binom{r-r_1}{r_2}\cdots\binom{r-r_1-\cdots r_{n-2}}{r_{n-1}}\cdot 1\\ &= \frac{r!}{r_1!(r-r_1)!}\cdot\frac{(r-r_1)!}{r_2!(r-r_1-r_2)!}\cdots\frac{(r-r_1-\cdots r_{n-2})!}{r_{n-1}!r_n!}\\ &= \frac{r!}{r_1!r_2!\cdots r_n!}\end{aligned} \tag{6.11}$$
Here's another way to see it. There are $r!$ permutations of r objects. Because the ordering in a cell doesn't matter, we divide by $r_1!$ to remove the reorderings in the first cell. Then, we divide by $r_2!$ to remove the reorderings in the second cell, and so on until we divide by $r_n!$ to remove the reorderings in the last cell.

   This quantity $\frac{r!}{r_1!r_2!\cdots r_n!}$ is called a *multinomial coefficient*, often abbreviated $\binom{r}{r_1\ r_2\ \cdots\ r_n}$. When $n=2$, then $r_2 = r - r_1$, and the multinomial coefficient reduces to the binomial coefficient $\binom{r}{r_1}$.

Example — Placing three balls into three cells

For a concrete example of these ideas, consider the problem of placing three balls, denoted a, b, and c, into three cells. We permit the balls to be distinguished, but do not care about the ordering of the balls within each cell. Thus, $a|bc|-$ is the same as $a|cb|-$ but different from $b|ac|-$ and $c|ab|-$. The possible arrangements are exhaustively listed in the accompanying table.

Table 1: Placing three balls into three cells

Indistinguishable balls (occupancy vectors)										
	$\frac{3!}{1!2!} = 3$				$\frac{3!}{1!1!1!} = 6$				$\frac{3!}{3!} = 1$	
$(3,0,0)$	$(0,3,0)$	$(0,0,3)$	$(2,1,0)$	$(2,0,1)$	$(0,2,1)$	$(1,2,0)$	$(1,0,2)$	$(0,1,2)$	$(1,1,1)$	10
$abc\vert-\vert-$	$-\vert abc\vert-$	$-\vert-\vert abc$	$ab\vert c\vert-$	$ab\vert-\vert c$	$-\vert ab\vert c$	$a\vert bc\vert-$	$a\vert-\vert bc$	$-\vert a\vert bc$	$a\vert b\vert c$	27
			$ac\vert b\vert-$	$ac\vert-\vert b$	$-\vert ac\vert b$	$b\vert ac\vert-$	$b\vert-\vert ac$	$-\vert b\vert ac$	$a\vert c\vert b$	
			$bc\vert a\vert-$	$bc\vert-\vert a$	$-\vert bc\vert a$	$c\vert ab\vert-$	$c\vert-\vert ab$	$-\vert c\vert ab$	$b\vert a\vert c$	
									$b\vert c\vert a$	
									$c\vert a\vert b$	
									$c\vert b\vert a$	
	$\frac{3!}{3!0!0!} = 1$				$\frac{3!}{2!1!0!} = 3$				$\frac{3!}{1!1!1!} = 6$	
Distinguishable balls (arrangements)										

Here is how the various formulas apply to this problem. Not distinguishing the balls but only how many are in each cell (*i.e.*, the balls are treated as bosons) results in a set of "occupancy vectors": $\{(3,0,0),(0,3,0),\ldots,(1,1,1)\}$. The number of distinct occupancy vectors is how many ways you can arrange three balls and two walls,

$$\binom{n+r-1}{r} = \binom{5}{3} = 10 \ .$$

These ten distinct occupancies are accounted for as follows. The number of occupancy vectors with the numbers $\{3,0,0\}$ in any order is given by a multinomial coefficient,

$$\frac{3! \text{ permutations of 3 numbers}}{(1! \text{ permutation of '3'}) \cdot (2! \text{ permutations of '0'})} = \frac{3!}{2!} = 3 \ .$$

Similarly, the number of occupancy vectors with the numbers $\{0,1,2\}$ in any order is

$$\frac{3!}{1! \cdot 1! \cdot 1!} = 3! = 6$$

and the number of occupancy vectors with $\{1,1,1\}$ in any order is

$$\frac{3!}{3!} = 1 \ .$$

These add up to ten.

Now we consider the distinguishable arrangements of the balls (*i.e.*, the balls are treated as Maxwell particles). The number of ways to place r balls in n cells is $n^r = 3^3 = 27$. These are accounted as follows. In the first column of the table, there are three occupancy vectors with $\frac{3!}{3! \cdot 0! \cdot 0!} = 1$ distinguishable arrangement each (again, a multinomial coefficient). In the second column, there are six occupancy vectors with $\frac{3!}{2! \cdot 1! \cdot 0!} = 3$ arrangements each, and in the last column, there is only one occupancy vector but $\frac{3!}{1! \cdot 1! \cdot 1!} = 6$ arrangements. The total number of arrangements, therefore, is

$$\sum_{\text{all occupancies}} \# \text{ arrangements per occupancy vector} = 3 \cdot 1 + 6 \cdot 3 + 1 \cdot 6 = 27 \ .$$

6.7. Summary

- Sum rule: If A and B are disjoint, the number of ways to do A or B is $|A| + |B|$.

- Product rule: The number of ways to do A followed by B is $|A| \cdot |B|$.

- Permutations: The number of r-element *ordered* arrangements of n distinct items is $P_r^n = \dfrac{n!}{(n-r)!}$. "How many ways can I *arrange*?"

- Combinations: The number of r-element *unordered* arrangements of n distinct items is $C_r^n = \dbinom{n}{r} = \dfrac{n!}{r!(n-r)!}$. "How many ways can I *choose*?"

- Maxwell-Boltzmann statistics: r identical, distinguishable objects can be assigned to n cells in n^r ways.

- Bose-Einstein statistics: r identical, *in*distinguishable objects can be assigned to n cells in $C_{n-1}^{n+r-1} = \binom{n+r-1}{n-1} = \binom{n+r-1}{r}$ ways if there is no limit on the population of each cell.

- Fermi-Dirac statistics: r identical, *in*distinguishable objects can be assigned to n cells with only one object per cell ($r \leq n$) in $C_{n-r}^{n} = \binom{n}{n-r} = \binom{n}{r}$ ways.

- Multinomial statistics: r identical, *in*distinguishable objects can be assigned to n cells with r_k objects in the k^{th} cell, in $\binom{r}{r_1\ r_2\ \cdots\ r_n} = \frac{r!}{r_1!r_2!\cdots r_n!}$ ways.