Unconditionally Stable Finite Difference Time Domain Method (ADI – FDTD)

Maxwell's Equations:

$$\nabla \times \vec{H} = s\,\vec{E} + ee_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = -mm_0 \frac{\partial \vec{H}}{\partial t}\ ;$$

Can be split into 6 scalar equations:

$$s\,E_x + ee_0 \frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}$$

$$s\,E_y + ee_0 \frac{\partial E_y}{\partial t} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}$$

$$s\,E_z + ee_0 \frac{\partial E_z}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}$$

$$mm_0 \frac{\partial H_x}{\partial t} = \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}$$

$$mm_0 \frac{\partial H_y}{\partial t} = \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}$$

$$mm_0 \frac{\partial H_z}{\partial t} = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}$$

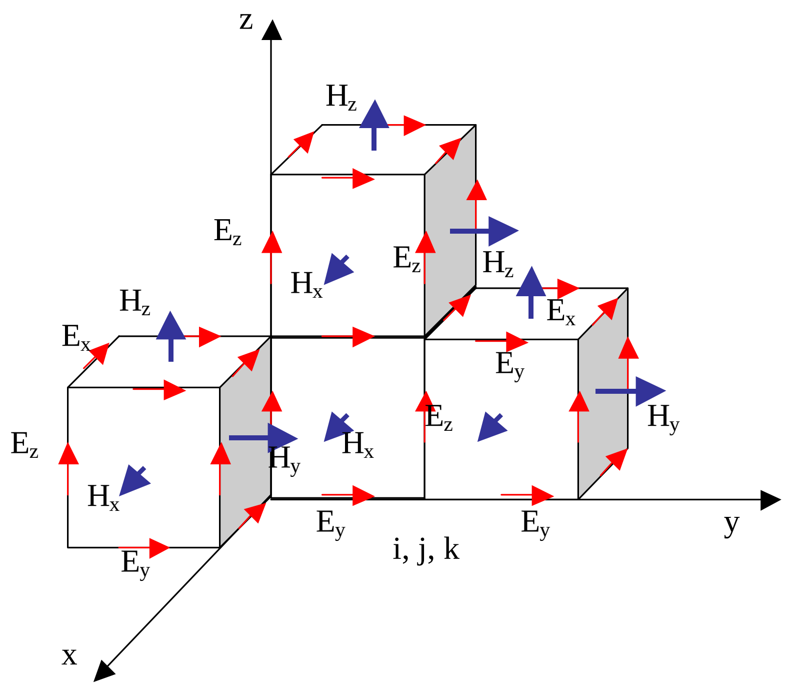


Fig 1. FD-TD Space scheme.

Denote any component of the fields

$F_a\big|_{i,j\,k}^{n} = F_a\,(n\Delta t, i\Delta x,\ j\Delta y, k\Delta z)$, where a =x,y or z, n, i, j,k are time and space indexes,

Δt is the time step, $\Delta x, \Delta y, \Delta z$ *are* the spatial increment steps along x,y,z directions, respectively.

ADI FD-TD

The (n+1)-th time step is broken up into two computational sub-advancements: the advancement from the n-th time step to the (n+1/2) time step and the advancement from the (n+1/2) time step to the (n+1)th time step.

First procedure : n+1/2-th time step:

$$E_x^{n+1/2}(i+\frac{1}{2}, j,k) = C_a(i+\frac{1}{2}, j,k)E_x^n(i+\frac{1}{2}, j,k) + C_b(i+\frac{1}{2}, j,k)$$
$$\cdot\left[\frac{\left\{H_z^n(i+\frac{1}{2}, j+\frac{1}{2},k) - H_z^n(i+\frac{1}{2}, j-\frac{1}{2},k)\right\}}{dy(j)} - \frac{\left\{H_y^{n+1/2}(i+\frac{1}{2}, j,k+\frac{1}{2}) - H_y^{n+1/2}(i+\frac{1}{2}, j,k-\frac{1}{2})\right\}}{dz(k)}\right]$$

$$E_y^{n+1/2}(i, j+\frac{1}{2},k) = C_a(i, j+\frac{1}{2},k)E_y^n(i, j+\frac{1}{2},k) + C_b(i, j+\frac{1}{2},k)$$
$$\cdot\left[\frac{\left\{H_x^n(i, j+\frac{1}{2},k+\frac{1}{2}) - H_x^n(i, j+\frac{1}{2},k-\frac{1}{2})\right\}}{dz(k)} - \frac{\left\{H_z^{n+1/2}(i+\frac{1}{2}, j+\frac{1}{2},k) - H_z^{n+1/2}(i-\frac{1}{2}, j+\frac{1}{2},k)\right\}}{dx(i)}\right]$$

$$E_z^{n+1/2}(i, j,k+\frac{1}{2}) = C_a(i, j,k+\frac{1}{2})E_z^n(i, j,k+\frac{1}{2}) + C_b(i, j,k+\frac{1}{2})$$
$$\cdot\left[\frac{\left\{H_y^n(i+\frac{1}{2}, j,k+\frac{1}{2}) - H_y^n(i-\frac{1}{2}, j,k+\frac{1}{2})\right\}}{dx(i)} - \frac{\left\{H_x^{n+1/2}(i, j+\frac{1}{2},k+\frac{1}{2}) - H_z^{n+1/2}(i, j-\frac{1}{2},k+\frac{1}{2})\right\}}{dy(j)}\right]$$

$$H_x^{n+1/2}(i, j+\frac{1}{2}, k+\frac{1}{2}) = H_x^n(i, j+\frac{1}{2}, k+\frac{1}{2}) + D_b(i, j+\frac{1}{2}, k+\frac{1}{2})$$

$$\cdot\left[\frac{\left\{E_y^n(i, j+\frac{1}{2}, k+1) - E_y^n(i, j+\frac{1}{2}, k)\right\}}{dz(k)} - \frac{\left\{E_z^{n+1/2}(i, j+1, k+\frac{1}{2}) - E_z^{n+1/2}(i, j, k+\frac{1}{2})\right\}}{dy(j)}\right]$$

$$H_y^{n+1/2}(i+\frac{1}{2}, j, k+\frac{1}{2}) = H_y^n(i+\frac{1}{2}, j, k+\frac{1}{2}) + D_b(i+\frac{1}{2}, j, k+\frac{1}{2})$$

$$\cdot\left[\frac{\left\{E_z^n(i+1, j, k+\frac{1}{2}) - E_z^n(i, j, k+\frac{1}{2})\right\}}{dx(i)} - \frac{\left\{E_x^{n+1/2}(i+\frac{1}{2}, j, k+1) - E_x^{n+1/2}(i+\frac{1}{2}, j, k)\right\}}{dz(k)}\right]$$

$$H_z^{n+1/2}(i+\frac{1}{2}, j+\frac{1}{2}, k) = H_z^n(i+\frac{1}{2}, j+\frac{1}{2}, k) + D_b(i+\frac{1}{2}, j+\frac{1}{2}, k)$$

$$\cdot\left[\frac{\left\{E_x^n(i+\frac{1}{2}, j+1, k) - E_x^n(i+\frac{1}{2}, j, k)\right\}}{dy(j)} - \frac{\left\{E_y^{n+1/2}(i+1, j+\frac{1}{2}, k) - E_y^{n+1/2}(i, j+\frac{1}{2}, k)\right\}}{dx(i)}\right]$$

Second Procedure

$$E_x^{n+1}(i+\frac{1}{2}, j,k) = C_a(i+\frac{1}{2}, j,k)E_x^{n+1/2}(i+\frac{1}{2}, j,k) + C_b(i+\frac{1}{2}, j,k)$$

$$\cdot\left[\frac{\left\{H_z^{n+1}(i+\frac{1}{2}, j+\frac{1}{2}, k) - H_z^{n+1}(i+\frac{1}{2}, j-\frac{1}{2}, k)\right\}}{dy(j)} - \frac{\left\{H_y^{n+1/2}(i+\frac{1}{2}, j, k+\frac{1}{2}) - H_y^{n+1/2}(i+\frac{1}{2}, j, k-\frac{1}{2})\right\}}{dz(k)}\right]$$

$$E_y^{n+1}(i, j+\frac{1}{2}, k) = C_a(i, j+\frac{1}{2}, k)E_y^{n+1/2}(i, j+\frac{1}{2}, k) + C_b(i, j+\frac{1}{2}, k)$$

$$\cdot\left[\frac{\left\{H_x^{n+1}(i, j+\frac{1}{2}, k+\frac{1}{2}) - H_x^{n+1}(i, j+\frac{1}{2}, k-\frac{1}{2})\right\}}{dz(k)} - \frac{\left\{H_z^{n+1/2}(i+\frac{1}{2}, j+\frac{1}{2}, k) - H_z^{n+1/2}(i-\frac{1}{2}, j+\frac{1}{2}, k)\right\}}{dx(i)}\right]$$

$$E_z^{n+1}(i, j, k+\frac{1}{2}) = C_a(i, j, k+\frac{1}{2})E_z^{n+1/2}(i, j, k+\frac{1}{2}) + C_b(i, j, k+\frac{1}{2})$$

$$\cdot\left[\frac{\left\{H_y^{n+1}(i+\frac{1}{2}, j, k+\frac{1}{2}) - H_y^{n+1}(i-\frac{1}{2}, j, k+\frac{1}{2})\right\}}{dx(i)} - \frac{\left\{H_x^{n+1/2}(i, j+\frac{1}{2}, k+\frac{1}{2}) - H_z^{n+1/2}(i, j-\frac{1}{2}, k+\frac{1}{2})\right\}}{dy(j)}\right]$$

$$H_x^{n+1}(i, j+\frac{1}{2}, k+\frac{1}{2}) = H_x^{n+1/2}(i, j+\frac{1}{2}, k+\frac{1}{2}) + D_b(i, j+\frac{1}{2}, k+\frac{1}{2})$$

$$\cdot\left[\frac{\left\{E_y^{n+1}(i, j+\frac{1}{2}, k+1) - E_y^{n+1}(i, j+\frac{1}{2}, k)\right\}}{dz(k)} - \frac{\left\{E_z^{n+1/2}(i, j+1, k+\frac{1}{2}) - E_z^{n+1/2}(i, j, k+\frac{1}{2})\right\}}{dy(j)}\right]$$

$$H_y^{n+1}(i+\frac{1}{2}, j, k+\frac{1}{2}) = H_y^{n+1/2}(i+\frac{1}{2}, j, k+\frac{1}{2}) + D_b(i+\frac{1}{2}, j, k+\frac{1}{2})$$

$$\cdot\left[\frac{\left\{E_z^{n+1}(i+1, j, k+\frac{1}{2}) - E_z^{n+1}(i, j, k+\frac{1}{2})\right\}}{dx(i)} - \frac{\left\{E_x^{n+1/2}(i+\frac{1}{2}, j, k+1) - E_x^{n+1/2}(i+\frac{1}{2}, j, k)\right\}}{dz(k)}\right]$$

$$H_z^{n+1}(i+\frac{1}{2}, j+\frac{1}{2}, k) = H_z^{n+1/2}(i+\frac{1}{2}, j+\frac{1}{2}, k) + D_b(i+\frac{1}{2}, j+\frac{1}{2}, k)$$

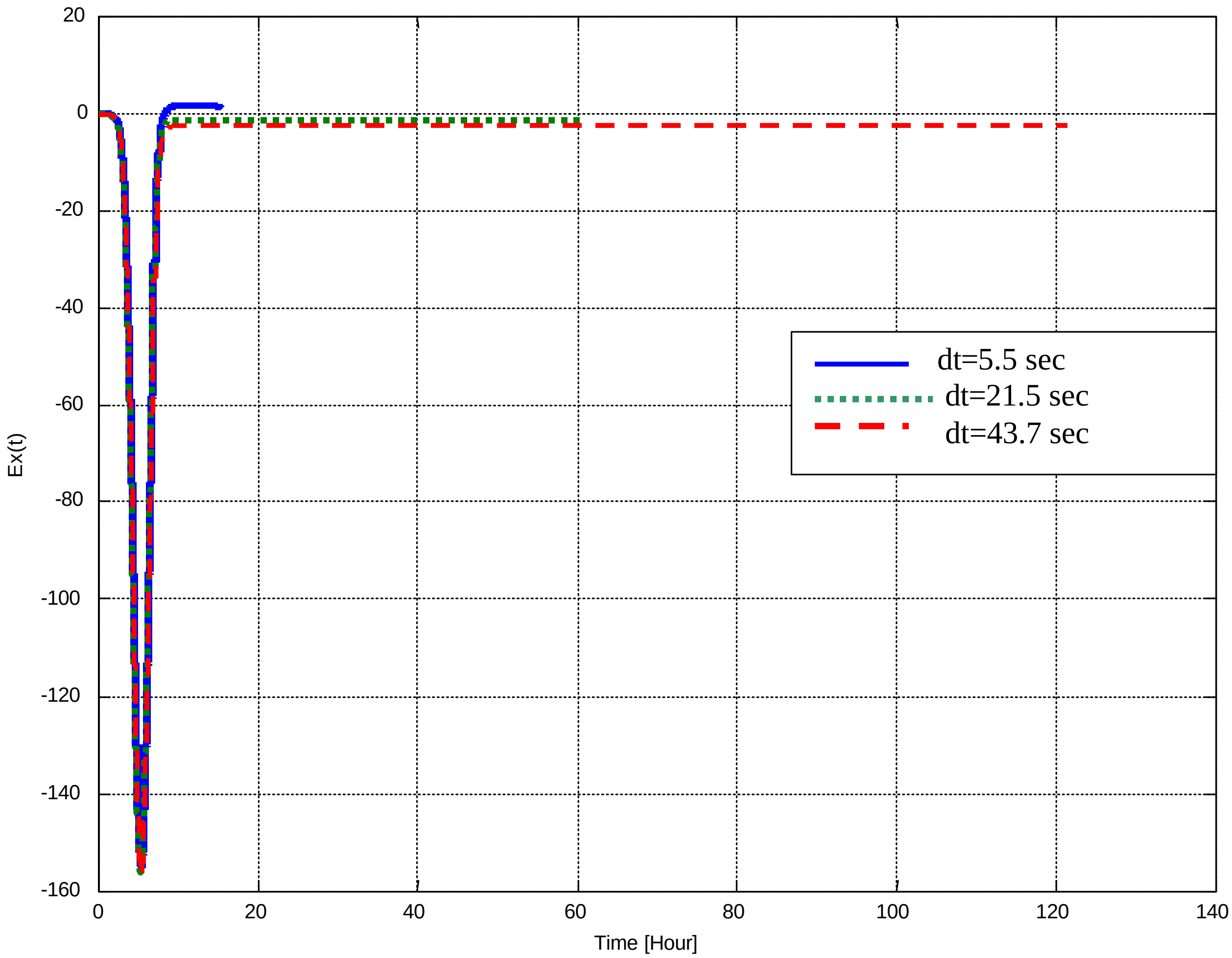
$$\cdot\left[\frac{\left\{E_x^{n+1}(i+\frac{1}{2}, j+1, k) - E_x^{n+1}(i+\frac{1}{2}, j, k)\right\}}{dy(j)} - \frac{\left\{E_y^{n+1/2}(i+1, j+\frac{1}{2}, k) - E_y^{n+1/2}(i, j+\frac{1}{2}, k)\right\}}{dx(i)}\right]$$

where

$$C_a(i, j, k) = \frac{2e_0 e(i, j, k) - s\,(i, j, k)\Delta t}{2e_0 e(i, j, k) + s\,(i, j, k)\Delta t}$$

$$C_a(i, j, k) = \frac{2\Delta t}{2e_0 e(i, j, k) + s\,(i, j, k)\Delta t}$$

$$D_b(i, j, k) = \frac{\Delta t}{m_0 m(i, j, k)}$$



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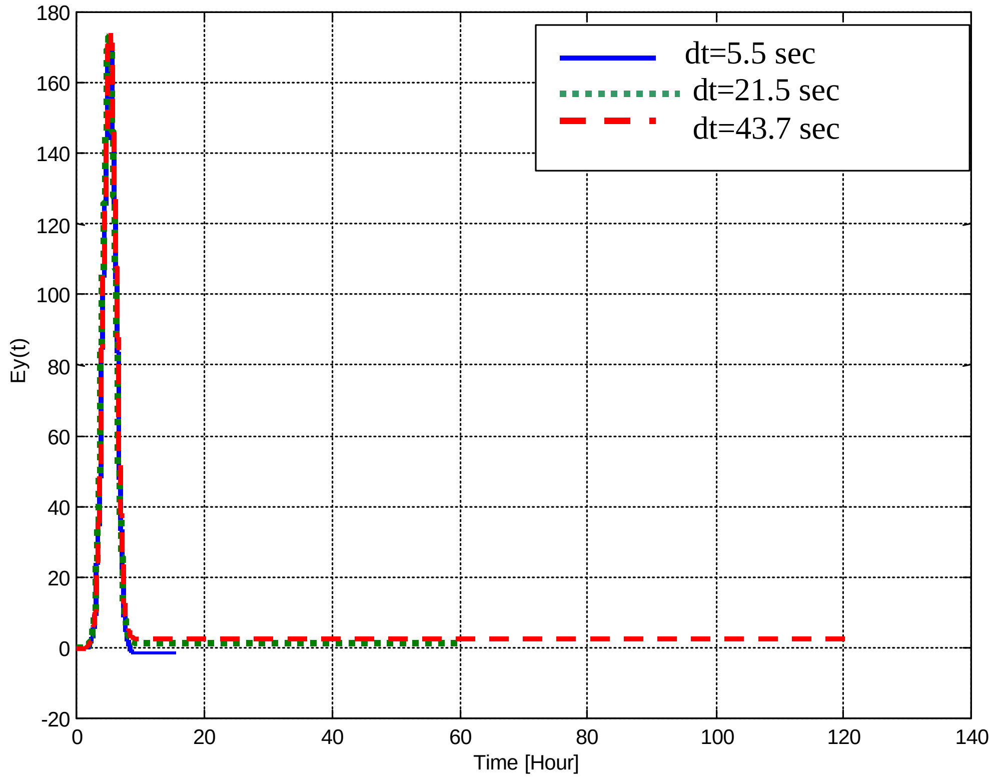

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