Environ Fluid Mech
DOI 10.1007/s10652-008-9082-7

ORIGINAL PAPER

Beyond eddy diffusivity: an alternative model for turbulent dispersion

Benoit Cushman-Roisin

Received: 8 May 2008 / Accepted: 5 July 2008

Abstract Turbulent dispersion proceeds not only much faster but also in a qualitatively different manner than molecular diffusion. Yet, the majority of hydraulic, oceanic and atmospheric models rely on the concept of an eddy diffusivity. It is shown here that an alternative model can be developed to exhibit observed behavior. The new term in the diffusion equation, which is non-local, may be interpreted in terms of the probability density function (pdf) of the turbulent velocity. Different assumptions about this distribution lead to a family of models, one of which is the model proposed here and another, the classical Fickian model of diffusion. A connection is also made with models using fractional calculus.

Keywords Turbulence · Dispersion · Fractional calculus

1 The problem with Fickian diffusion

Fick [6] was first in proposing a formula for molecular diffusion, whereby the flux of substance is taken proportional to its concentration gradient, with the coefficient of proportionality being called the diffusivity. The situation is different in turbulent flow, but an eddy diffusivity,[1] which is no more than an enlarged molecular diffusivity, is often invoked, and the flux q of concentration of a substance in the x-direction is still taken proportional to the gradient $\partial c/\partial x$ of the concentration $c(x, t)$ of the substance:

$$q = -D\frac{\partial c}{\partial x}, \tag{1}$$

where the coefficient D is the eddy diffusivity.

[1] The concept of an eddy diffusivity can be traced to [3] as an expedient way to overcome the turbulence closure problem exposed a few years earlier by Osborne Reynolds.

B. Cushman-Roisin (✉)
Thayer School of Engineering, Dartmouth College, Hanover, NH 03755, USA
e-mail: Benoit.Cushman.Roisin@dartmouth.edu; Benoit.R.Roisin@dartmouth.edu

Fig. 1 Aerial view of the turbulent dispersion of a smokestack plume. Here advection by wind makes downwind distance play the role of time so that a single snapshot traces the temporal evolution of the cross-wind spread. Note that the plume widens as a triangle, implying that the cross-wind width increase linearly with downwind distance and hence time. (Excerpted from "Blue Vinyl", documentary NVG-9683 directed by Judith Helfand and Daniel B. Gold, docurama.com ©2005 New Video Group, Inc.)

In combination with the mass budget for the substance, the previous representation yields the well known parabolic equation of diffusion, which in one dimension takes the form:

$$\frac{\partial c}{\partial t} = -\frac{\partial q}{\partial x} \longrightarrow \frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}. \tag{2}$$

This is called Fick's second law.

Fickian diffusion predicts that the length scale σ of a spreading patch increases with the square root of time ($\sigma^2 = 2Dt$). It has been argued for a long time, however, that the width of a dispersing patch in turbulent regime ought to grow faster than the square root of time [2,7]. Stommel [17] concluded that the Fickian model fails to describe horizontal diffusion in the sea, and [12] made an effort to find a new law of diffusion but fell short of doing so, proposing instead to invoke a variable eddy diffusivity, a widely used recipe in hydraulic and oceanographic modeling to this day. Countless observations in both water (ex. [4]) and air (ex. [10]) reveal patch sizes that grow roughly linearly with time. Figure 1 shows an unmistakable example in the atmosphere. This evidence leads to the conclusion that Fickian diffusion, no matter what value is ascribed to D, is fundamentally unable to reproduce turbulent dispersion.

Various approaches have been proposed to overcome this handicap. Among these, we ought to discard methods in which the diffusivity is made to depend explicitly on time or space because they fail as soon as multiple or continuous sources are present. Not falling in this group and showing much promise are methods using fractional calculus ([9], and references therein). Worth noting also is the review and discussion by [1]. Here, a simple and effective approach is proposed. In the jargon of fractional calculus, the new dispersion term can be interpreted as the square root of the second derivative.

2 A remedy

A heuristic approach to obtain a formulation that predicts spreading at a rate proportional to t instead of $t^{1/2}$ goes as follows: to ensure a length scale or patch width σ growing proportionally to t, the Fickian relation $\sigma^2 = 2Dt$ demands that the underlying diffusivity D be proportional to t and thus to σ; in other words, the diffusivity cannot remain constant but needs to grow with the patch size. Physically, this means that, of all eddy scales that exist

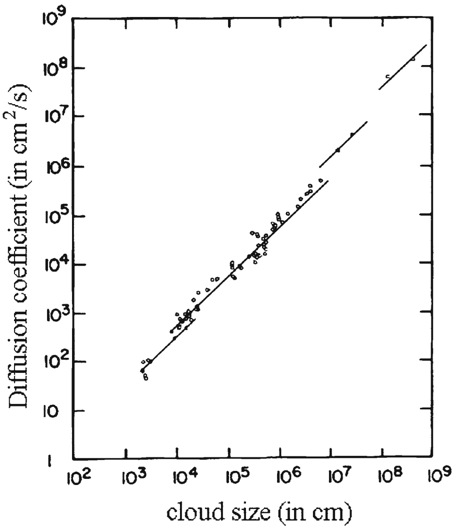

Fig. 2 Diffusion coefficient as function of patch size, according to [13]. Although a 4/3 power law is often fitted to this data set, closer inspection reveals that data groups better obey a law of simple proportionality, as shown here, with the coefficient of proportionality presumably increasing with the level of energy

simultaneously and among which the patch lies, those at the scale most comparable to the size of the patch are the most effective at distorting and dispersing it, letting the smaller eddies to smear details and the larger eddies to transport the patch. Such proportionality between diffusivity and size has been observed in the ocean over a broad spectrum of length scales ([13], Fig. 2).

In the Fourier space, with k being the wavenumber associated with the patch's length scale, D must be inversely proportional to k, and we write

$$D = \frac{u_*}{|k|}, \tag{3}$$

in which the coefficient u_* has the dimension of a velocity and can be interpreted as a turbulent velocity (proportional to the friction velocity in wall turbulence or to the square root of the turbulent kinetic energy in homogeneous turbulence).

The budget Eq. 2 written in Fourier space is

$$\frac{\partial \hat{c}}{\partial t} = -Dk^2 \hat{c}, \tag{4}$$

in which $\hat{c}$ is the Fourier transform of c, and with (3), becomes

$$\frac{\partial \hat{c}}{\partial t} = -u_* |k| \hat{c}. \tag{5}$$

Taking the reverse Fourier transform then brings about a new equation in space-time:

$$\frac{\partial c}{\partial t} = \frac{u_*}{\pi} \int_{-\infty}^{+\infty} \frac{c(x-\xi, t) - c(x, t)}{\xi^2} d\xi, \tag{6}$$

in which ξ is a dummy variable. This equation, which supersedes (2), is guaranteed to produce a patch size that grows proportionally to time. This equation is autonomous (no coefficient depends explicitly on time or distance), as it should be, and it is trivial to show that it conserves the spatial integral of c (conservation of tracer mass). Note how the diffusivity D has been superseded by the velocity u_* as the parameter measuring the strength of the dispersion.

The spatial operator on the right-hand side is non-local, fulfilling the expectation of [2], who wrote (p 360) "Turbulent diffusion is a non local effect (...), and a description of the diffusion with some kind of integral equation is more to be expected." There is also a similarity with the non-local term used to represent turbulent mixing of momentum in [5].

3 A more rigorous derivation

The preceding derivation of (6) was heuristic and contrived by the expected behavior of its solution. The same equation, however, can be derived from basic principles. Turbulent dispersion being essentially random advection, the underlying equation governing the concentration $c(x, t)$ of a substance is, in one dimension:

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = 0, \tag{7}$$

with the velocity u being a random variable. Such equation is traditionally first subjected to a Reynolds decomposition between mean and fluctuating components ($c = \bar{c} + c'$ and $u = \bar{u} + u'$) and then averaged over the fluctuations. A solution, however, cannot be found until the Reynolds flux ($\overline{u'c'}$) is expressed in terms of known quantities. Here, we circumvent the closure problem by reversing the procedure: we first solve the equation, with the fluctuations included, and then average its solution.

The solution of (7) over a short interval of time Δt is

$$c(x, t + \Delta t) = c(x - u\Delta t, t), \tag{8}$$

a mere translation from position $x - u\Delta t$ to x. An ensemble average over the turbulent fluctuations of u then yields

$$c(x, t + \Delta t) \; = \; \int_{-\infty}^{+\infty} c(x - u\Delta t, t) \; f(u) \; du, \tag{9}$$

in which $f(u)du$ is the probability that the turbulent velocity u lies in the interval $[u, u+du]$. For convenience, we shall assume here and throughout that u has a zero mean. Physically, this implies the absence of mean advection, which can be trivially added to the formalism.

Since the integral of $f(u)du$ over all values of u must be unity (normalization of probabilities), the preceding equation may be recast in terms of a difference over time

$$\frac{c(x, t + \Delta t) - c(x, t)}{\Delta t} = \int_{-\infty}^{+\infty} \frac{c(x - u\Delta t, t) - c(x, t)}{\Delta t} f(u)du, \tag{10}$$

after a further division by Δt. A switch from velocity u to displacement $\xi = u\Delta t$ then yields:

$$\frac{c(x, t + \Delta t) - c(x, t)}{\Delta t} = \int_{-\infty}^{+\infty} \frac{c(x - \xi, t) - c(x, t)}{\Delta t} g(\xi, \Delta t)d\xi, \tag{11}$$

in which $g(\xi, \Delta t)d\xi$ is the probability of a jump of length ranging between ξ and $\xi + d\xi$ in time interval Δt. Immediate properties of this probability density distribution are:

$$g(\xi, \Delta t)d\xi \; = \; f(u)du \tag{12}$$

$$\int_{-\infty}^{+\infty} g(\xi, \Delta t)d\xi \; = 1 \tag{13}$$

$$\int_{-\infty}^{+\infty} \xi g(\xi, \Delta t)d\xi \; = 0. \tag{14}$$

The last of these statements, namely that the averaged jump length is nil, simply reflects the choice of zero mean advection.

The crux of the analysis consists in requiring at this point that the jump probability distribution obey the principle of divisibility, namely that the probability of a single jump over a given time interval follow the same distribution as that of a combined set of two jumps over two half time intervals. This is required because the time interval Δt is arbitrary, and the limit of vanishing Δt may not be degenerate. The principle demands

$$g(\xi, 2\Delta t) = \int_{-\infty}^{+\infty} g(\xi', \Delta t) g(\xi - \xi', \Delta t) d\xi' \tag{15}$$

stating that the probability of landing at distance ξ after time $2\Delta t$ is equal to the probability of making a first jump ξ' during a first time interval Δt followed by a complementary jump of length $\xi - \xi'$ during a second time interval Δt, for all possible values of ξ'. Divisibility requires that the g function on the right be the same as that on the left. Note that divisibility is not identical to the notion of continuous-time random walk introduced by [8]. The latter considered continuous time in discrete space whereas we consider continuity of both space and time.

Because the function g has the inverse dimension of a length, the time dimension of Δt must somehow be canceled by the time dimension of a physical variable. In other words, there must exist at least one physical parameter in the formalism that permits the construction of a dimensionless ratio. While a number of possibilities can be invoked, a single-parameter approach seems to hold the greatest appeal, for the use of multiple parameters can most likely be reducible to a combination of single-parameter cases.

Using the turbulent velocity scale u_* to form the dimensionless ratio $a = \xi / u_* \Delta t$, the function g can be expressed as

$$g(\xi, \Delta t) = \frac{1}{u_* \Delta t} G(a), \tag{16}$$

and the divisibility requirement becomes:

$$\frac{1}{2} G\left(\frac{a}{2}\right) = \int_{-\infty}^{+\infty} G(a') G(a - a') da' \tag{17}$$

of which the solution is

$$G(a) = \frac{1}{\pi} \frac{A}{a^2 + A^2}, \tag{18}$$

in which A is an arbitrary dimensionless constant. This function is known as the Cauchy probability distribution function (pdf). Returning to the dimensional function g, we have

$$g(\xi, \Delta t) = \frac{1}{\pi} \frac{A u_* \Delta t}{\xi^2 + (A u_* \Delta t)^2}. \tag{19}$$

Substituting pdf (19) in (11) and taking the limit $\Delta t \to 0$ yields

$$\frac{\partial c}{\partial t} = \frac{A u_*}{\pi} \int_{-\infty}^{+\infty} \frac{c(x - \xi, t) - c(x, t)}{\xi^2} d\xi, \tag{20}$$

We recover here Eq. 6, except for the multiplicative constant A, which can be absorbed in a redefinition of the parameter u_*. The corresponding pdf for velocity is

$$f(u) = \frac{1}{\pi} \frac{A u_*}{u^2 + (A u_*)^2}. \tag{21}$$

4 Generalization and connection with fractional calculus

If instead of the velocity u_* (with dimension LT $^{-1}$) to tie ξ and Δt in a dimensionless construct $\xi/u_*\Delta t$, we had invoked a diffusivity D (with dimension L^2T^{-1}), the dimensionless ratio would have been $\xi/\sqrt{2D\Delta t}$, the divisibility condition been

$$\frac{1}{\sqrt{2}}G\left(\frac{a}{\sqrt{2}}\right) = \int_{-\infty}^{+\infty} G(a')G(a-a')da' \tag{22}$$

and the corresponding probability distribution function for jumps been

$$g(\xi, \Delta t) = \frac{1}{\sqrt{4\pi D\Delta t}} \exp\left(-\frac{\xi^2}{4D\Delta t}\right). \tag{23}$$

Substitution in (11) and the limit $\Delta t \to 0$ would then have led to a recovery of Fickian diffusion (2). In other words, the resulting model depends on the dimensional quantity of choice.

In general, if we choose a quantity Q of dimension $LT^{-\alpha}$, the jump probability distribution must take the form

$$g(\xi, \Delta t) = \frac{1}{Q\Delta t^\alpha} G\left(\frac{\xi}{Q\Delta t^\alpha}\right), \tag{24}$$

with the dimensionless function G obeying the divisibility requirement

$$\frac{1}{2^\alpha}G\left(\frac{a}{2^\alpha}\right) = \int_{-\infty}^{+\infty} G(a')G(a-a')da' \tag{25}$$

A continuous series of models can be generated, each with its own dispersive behavior. The case $\alpha = 1/2$ returns classical Fickian diffusion, whereas $\alpha = 1$ yields the model proposed here. It can be shown that the general case gives an equation with operator of fractional order $1/\alpha$ and patch size increasing like t^α. Batchelor and Townsend [2] advocated $t^{3/2}$ as the growth rate for turbulent dispersion in homogeneous turbulence. The corresponding value $\alpha = 3/2$ is to be associated with a physical quantity of dimensions $LT^{-3/2}$, or power thereof. The underlying dimensional quantity is the rate of energy dissipation ϵ (with dimension L^2T^{-3}) of Kolmogorov's inertial subrange, via the so-called Richardson–Obukhov law [11,14,15]:

$$\sigma^2 = \text{constant}\, \epsilon\, t^3. \tag{26}$$

Note that this relation also corresponds to the case of an eddy diffusivity proportional to the 4/3 power of the length scale [13,16].

Although all intermediate possibilities are mathematically realizable, it seems difficult to justify on physical grounds the use of dimensional quantities other than canonical ones such as molecular diffusivity D ($L^2T^{-1}, \alpha = 1/2$), turbulent velocity u_* ($L^{-1}, \alpha = 1$), and energy dissipation ϵ ($L^2T^{-3}, \alpha = 3/2$). It can be presumed that evidence of any other value may be the apparent outcome of a combination of several of the primary physical variables.

5 Conclusions

A new equation with a non-local term, namely (6), is proposed for turbulent dispersion in preference to classical Fickian diffusion. The key physical parameter is not a diffusivity but a turbulent velocity u_*, the value of which is readily assigned in any particular situation.

Predicted spreading is proportional to time, not to its square root, in agreement with common observations.

Acknowledgements This work was supported by grant N00014-02-1-0065 from the U.S. Office of Naval Research to Dartmouth College.

References

1. Bakunin OG (2004) Correlation effects and turbulent diffusion scalings. Rep Prog Phys 67:965–1032
2. Batchelor GK, Townsend AA (1956) Turbulent diffusion. In: Batchelor GK, Davies RM (eds) Surveys in mechanics. Cambridge University Press, pp 352–399
3. Boussinesq J (1877) Essai sur la thèorie des eaux courantes [Essay on the theory of flowing waters]. Mem Acadèmie des Sciences, vol 23. Inst France, Paris, pp 252–260
4. Clark JF, Schlosser P, Stute M, Simpson HJ (1996) $SF_6-{}^3He$ tracer release experiment: a new method of determining longitudinal dispersion coefficients in large rivers. Environ Sci Technol 30:1527–1532
5. Cushman-Roisin B, Jenkins AD (2006) On a non-local parameterization for shear turbulence and the uniqueness of its solutions. Boundary-Layer Meteor 118:69–82
6. Fick A (1855) On liquid diffusion. Philos Mag J Sci 10:31–19
7. Gifford FA (1957) Relative atmospheric diffusion of smoke puffs. J Meteor 14:410–414
8. Klafter J, Blumen A, Schlesinger MF (1987) Stochastic pathway to anomalous diffusion. Phys Rev A 35:3081–3085
9. Meerschaert MM, Mortensen J, Wheatcraft SW (2006) Fractional vector calculus for fractional advection-dispersion. Physica A 367:181–190
10. Min IA, Abernathy RN, Lundblad HL (2002) Measurement and analysis of puff dispersion above the atmospheric boundary layer using quantitative imagery. J Appl Meteor 41:1027–1041
11. Obukhov AM (1941) Spectral energy distribution in turbulent flow. Izv Akad Nauk SSSR 5:453–566
12. Okubo A (1962) A review of theoretical models for turbulent diffusion in the sea. J Oceanogr Soc Jpn 20th Anniversary Volume:286–320
13. Okubo A (1974) Some speculations on oceanic diffusion diagrams. Rapports et Procès-verbaux des Rèunions du Conseil Permanent International pour l'Exploration de la Mer 167:77–85
14. Ouellette NT, Xu H, Bourgoin M, Bodenschatz E (2006) An experimental study of turbulent relative dispersion models. New J Phys 8. doi:10.1088/1367-2630/8/6/109.
15. Richardson LF (1926) Atmospheric diffusion shown on a distance-neighbour graph. Proc R Soc London A 110:709–737
16. Richardson LF, Stommel H (1948) A note on eddy diffusion in the sea. J Meteor 5:238–240
17. Stommel H (1949) Horizontal diffusion due to oceanic turbulence. J Mar Res 8:199–225