ENGS-43 – Winter 2012
ENVIRONMENTAL TRANSPORT & FATE

HOMEWORK #3 – SOLUTIONS

1. (10 points) Along the James River estuary in Virginia, the salinity *S* of the water has been measured (in parts per million, ppm) versus upstream distance from the mouth of the river ($x = 0$), and the data are:

Upstream distance *x*	Measured salinity *S*
far upstream	0 ppm
-27 miles	5,000 ppm
-15 miles	9,300 ppm
-10 miles	12,000 ppm
-5 miles	15,500 ppm
0 (mouth)	20,000 ppm

Assuming that this salinity distribution reflects a balance between upstream diffusion from the ocean and downstream advection by the tidally-averaged river flow u = 3.35 cm/s, estimate the longitudinal diffusivity D (in m^2/s).

The first thing to do is to plot these data and see what kind of distribution they generate (not done here in plain text, unfortunately). The plot reveals a monotonic rise of S with x, with a moderate upward curvature. Looking backwards, namely starting with the mouth ($x = 0$) and proceeding upstream (x increasingly negative), one notes an asymptotic approach to $S = 0$.

The processes at work are, as stated, advection and diffusion, both in the x-direction. Thus, the governing equation is:

$$u \frac{dS}{dx} = D \frac{d^2 S}{dx^2} ,$$

of which the most general solution is (with A and B being two constants of integration):

$$S(x) = A \exp\left(+ \frac{ux}{D} \right) + B .$$

Since far upstream ($x \to -\infty$), the salinity goes to zero, the constant B must be zero. To determine the remaining parameters of the solution, the constant A and the diffusivity D, we fit the $S(x)$ profile to the data. There are 5 data points but only two unknowns. Therefore, the problem is overdetermined and the best we can do is to find a fit that minimizes the discrepancy between actual data and curve prediction.

The best method for such fit is the least-square technique, which is most handy when we try to fit a straight line (linear relationship) between data and curve. To turn the curve into a straight line, we take the natural logarithm of the function:

$$\ln[S(x)] = \ln A + \frac{u}{D} x ,$$

so that the logarithm of S is a linear function of x. The data to fit are then:

x (km)	S (ppm)
-43	5,000
-24	9,300
-16	12,000
-8	15,500
0	20,000

Linear regression (= least-square fit of a straight line, by minimizing the sum of the squares of the differences between data and straight line) yields:

$$\ln[S(x) \; in \;\; ppm] = 2.999 + 0.0322 \; x \; (in \; km)$$

The first term can be used to determine the constant A, but we are only interested in determining D. So, we only consider the second coefficient and fit it to the previous algebraic expression (watch for units!):

$$\frac{u}{D} = 0.0322 / km = 3.22 \times 10^{-5} \; / m .$$

Since the river velocity is known (u = 3.35 cm/s), we can solve for D:

$$\boxed{D = \frac{3.35 \times 10^{-2} \;\; m/s}{3.22 \times 10^{-5} \; / m} = 1{,}039 \; m^2 / s}.$$

2. (10 points) On the strike of midnight on a dark Friday in a valley near you, 270 kg of a toxic substance were released by a terrorist. In the turbulence of the explosion, the concentration very quickly reached homogeneity over the cross-section (= 30,000 m^2) of the valley. At the time of the accident and during the next four days, the wind was steadily blowing at a speed of 4 m/s along the valley. The toxic substance is non-conservative, decaying at the rate of 0.8/hour, and the diffusivity along the valley can be taken as uniform and constant, with a value of 1.25 m^2/s. If the toxicity level is 0.5 μg/L, how long did the toxicity episode last? And, which length of the valley was subjected to a toxic level at one time or another?

(**www.hamilton-media.co.uk/id6.html**)

This is a case of a one-dimensional instantaneous and punctual release with advection (wind) and decay. The solution for the concentration c as function of downwind distance x from the release and time t since the terrorist attack is

$$c(x,t) = \frac{M}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x-ut)^2}{4Dt} - Kt \right)$$

where

M = amount released / cross-sectional area = (270 kg)/(30,000 m^2) = 9.00 g/m^2,
u = wind speed = 4.0 m/s,
D = diffusivity coefficient = 1.25 m^2/s, and
K = decay rate = 0.8 / hour = 2.22 x 10^{-4} /s.

The concentration exhibits a bell-shape distribution that spreads and flattens over time while at the same time translating downwind.

To determine the extent of the toxic region, we look at its upwind and downwind edges where the concentration equals the toxic level of

$$c = 0.5 \text{ μg/L} = 0.5 \text{ mg/m}^3 = 5 \times 10^{-4} \text{ g/m}^3.$$

This sets a relationship between x and t in the solution. Solving for x in terms of t (because it is easier than the other round), we get:

$$x = ut \pm \sqrt{4Dt \left[\ln\left(\frac{M}{c\sqrt{4\pi Dt}} \right) - K\, t \right]}$$

The +/– sign (originating from a square root along the way) represents the fact that there is an upwind edge (– sign) and a downwind edge (+ sign), centered on $x = ut$ = position of the peak. Replacing the known quantities by their values, we obtain:

$$x = 4.000t \pm \sqrt{5.00t \left[\ln\left(\frac{4542}{\sqrt{t}} \right) - 2.222 \times 10^{-4}\, t \right]}$$

with t expressed in seconds and x obtained in meters, or

$$x = 14.4t \pm \sqrt{0.0180t \left[\ln\left(\frac{75.69}{\sqrt{t}} \right) - 0.800t \right]}$$

with t now expressed in hours and x obtained in kilometers.

Before trying multiple values of time, we note that at some point what is under the square-root will become negative. This occurs when the peak value of the concentration falls below the toxic value and there is no longer any region of the valley with toxic air. In other words, the end of the toxic episode corresponds to

$$\ln\left(\frac{75.69}{\sqrt{t_{end}}} \right) = 0.800t$$

$$t_{end} = 4.472 \; hours = 4 \; hours \; 28 \text{ min}$$

Thus, the **duration of the toxic episode is about 4 and half hours**.

For times before then, values of *x* can be calculated. A sample is tabulated below.

time t	upwind edge (in km)	downwind edge (in km)
10 sec	0.021	0.059
30 sec	0.088	0.152
1 min	0.196	0.284
10 min	2.276	2.524
20 min	4.634	4.966
30 min	7.004	7.396
1 hour	14.148	14.652
1.5 hours	21.319	21.881
2 hours	28.507	29.093
2.5 hours	35.710	36.290
3 hours	42.927	43.473
3.5 hours	50.162	50.638
4 hours	57.423	57.777
4.472 hours	64.393	64.401

From these values, we note that **the toxic region stretches from the location of the release to a distance of 64.4 km downwind**.

3. (10 points) In a river of depth H, the downstream velocity u typically varies with height z above the bottom as

$$u(z) = U\left(\frac{z}{H}\right)^{1/7}$$

where U is the surface value. With a good level of approximation, the vertical diffusivity D is equal to 0.0045 UH. The combination of velocity shear and vertical diffusion creates shear dispersion in the downstream direction, with effective diffusivity K. Through the seasonal cycle, the river variables are such that H quadruples and U simultaneously doubles between the summer low and the winter high flows. What is the corresponding range of K values? Estimates suffice (= you don't have to perform the integrals but you are welcome to do so if you have the courage.).

In this case, with $u_{\min} = u(z = 0) =$ and $u_{\max} = u(z = H) = U$, the total velocity difference is $\Delta U = u_{\max} - u_{\min} = U$. Then, using the approximation for K, namely

$$K = 0.0042\,\frac{H^2 \Delta U^2}{D}$$

and the given expression for the vertical diffusivity D, we obtain:

$$K = \frac{0.0042}{0.0045}\,\frac{H^2 U^2}{UH} = 0.933\,UH$$

Using subscript 1 for summer and 2 for winter, we have the relations:

$$H_2 = 4\,H_1 \quad \text{and} \quad U_2 = 2\,U_1$$

and the respective K values are:

$$K_1 = 0.933\,U_1 H_1$$
$$K_2 = 0.933\,U_2 H_2 = 0.933\,(2U_1)(4H_1) = 7.467\,U_1 H_1$$

The winter value for the shear-dispersion effective diffusivity is 8 times larger in winter than in summer.

4. (10 points) The mass of air in the atmosphere is 5.267×10^{18} kg, and the earth receives 1.75×10^{17} W from the sun globally. Assuming that half of the energy received from the sun is being dissipated in the atmosphere (the rest going to land and sea), estimate the rate of dissipation ε in the atmosphere. Using the Kolmogorov theory of turbulence, estimate also the smallest eddy scale in the air and its ratio to the largest scale (the earth's radius, equal to 6,371 km). The molecular viscosity of air at ambient temperature and pressure is $\nu = 1.51 \times 10^{-5}$ m^2/s.

First, we determine the amount of energy supplied by the sun to the earth. On a per-mass basis, this is 1.75×10^{17} J/s divided by 5.267×10^{18} kg = 0.0332 J/(kg.s) = 0.0332 m^2/s^3. Of this, only 50%, or 0.01661 m^2/s^3, goes into the atmosphere. Since the dissipation rate is equal to the rate of energy supply, we have:

$$\varepsilon = 0.01661 \text{ m}^2/\text{s}^3.$$

The smallest eddy scale is given by

$$\begin{aligned} d_{\min} &\approx \nu^{3/4}\varepsilon^{-1/4} \\ &\approx (1.51\times10^{-5} \text{ m}^2/\text{s})^{3/4}(0.01661\,\text{m}^2/\text{s}^3)^{-1/4} \\ &\approx 0.000675\,\text{m} = 0.675\,\text{mm}. \end{aligned}$$

Its ratio to the longest length scale is

$$\frac{d_{\min}}{d_{\max}} = \frac{0.000\,675\text{ m}}{6\,371\,000\text{ m}} = 1.06\times10^{-10}$$

That is a ratio of 10 billion to 1!

Additional considerations:

The orbital velocity of eddies having a 200-m diameter is $u_* = 0.95(\varepsilon d)^{1/3} = 1.418$ m/s, and the diffusivity generated by these eddies is found to be

$$D = \frac{du_*}{16\pi} = \frac{(200\text{ m})(1.418\text{ m/s})}{50.265} = 5.65\text{ m}^2/\text{s}.$$

The Reynolds number is

$$\text{Re} = \frac{UL}{\nu} = 2.0\times10^{13}$$