ENGS-43, Winter 2012
ENVIRONMENTAL TRANSPORT & FATE

HOMEWORK #2 – SOLUTIONS

1. (10 points) In an infinitely long one-dimensional system, two instantaneous releases of equal magnitude M occur at the same time (say $t = 0$) and at a distance L apart from each other (say $x = 0$ and $x = L$). What is the maximum concentration value reached over time at the middle point (at $x = L/2$)?

Release 1 at $t = 0$, at $x = 0$ and of amount M leads to a first component of solution:

$$c_1(x,t) = \frac{M}{\sqrt{4\pi Dt}} \exp\left[-\frac{x^2}{4Dt}\right]$$

Release 2 at same $t = 0$ but at $x = L$ of amount M leads to a second component of solution:

$$c_2(x,t) = \frac{M}{\sqrt{4\pi Dt}} \exp\left[-\frac{(x-L)^2}{4Dt}\right]$$

The total solution is the sum of the two:

$$c(x,t) = c_1 + c_2 = \frac{M}{\sqrt{4\pi Dt}} \left[\exp\left[-\frac{x^2}{4Dt}\right] + \exp\left[-\frac{(x-L)^2}{4Dt}\right]\right]$$

At the middle position $x = L/2$, the two releases contribute equally by symmetry, and we have:

$$c_{middle} = c(L/2,t) = 2\frac{M}{\sqrt{4\pi Dt}} \exp\left[-\frac{(L/2)^2}{4Dt}\right] = \frac{M}{\sqrt{\pi Dt}} \exp\left(-\frac{L^2}{16Dt}\right)$$

To obtain the maximum over time, we take the time derivative and set it to zero:

$$\frac{dc}{dt} = 0 \rightarrow -\frac{1}{2t}\frac{M}{\sqrt{\pi Dt}}\exp(\cdots) + \frac{M}{\sqrt{\pi Dt}}\frac{L^2}{16Dt^2}\exp(\cdots) = 0 \rightarrow -\frac{1}{2t} + \frac{L^2}{16Dt^2} = 0$$

The solution is:

$$t = \frac{L^2}{8D}$$

Substitution in the expression for c_{middle} yields:

$$c_{\max} = \frac{M}{\sqrt{\pi D\,(L^2/8D)}}\exp\left(-\frac{L^2}{16D\,(L^2/8D)}\right) = \frac{M}{L}\sqrt{\frac{8}{\pi}}\exp\left(-\frac{1}{2}\right) = 0.968\frac{M}{L}$$

2. (10 points) A small boat crosses Lake Bled to reach the center island where it intends to deliver a load of wood stain to restore some wood panels inside the small church there (see photo below). But, on its way, it accidentally spills 10 kg of this product over an area of 5 m^2, at a location where the depth is 9 m. Assuming that the wood stain mixes well with water and taking the vertical diffusivity equal to 0.015 m^2/s, determine the concentration of wood stain at mid-depth and at the bottom at selected times. Does the mid-depth value reach a maximum before reaching its ultimate value? Also, what is this ultimate value? Finally, by which time would you say the wood stain is well mixed over the vertical?

This is a case of one-dimensional (vertical) diffusion from an instantaneous and localized release, with the complications of two impermeable boundaries. These boundaries are the bottom (say at $z = 0$) and the water surface (say $z = H = 9$m). The solution involves a doubly infinite set of images, which for a release at the top end, as is the case here, becomes the same solution as for the example of the Chicago Ship Canal shown on page 49 of the course notes. The expressions for the mid-depth and bottom concentrations are given as:

$$c_{mid-depth}(x = \frac{H}{2}, t) = \frac{M}{\sqrt{\pi D t}}\left[\exp\left(-\frac{H^2}{16Dt}\right) + \exp\left(-\frac{9H^2}{16Dt}\right) + \exp\left(-\frac{25H^2}{16Dt}\right) + \cdots\right]$$

$$c_{bottom}(x = 0, t) = \frac{M}{\sqrt{\pi D t}}\left[2\exp\left(-\frac{H^2}{4Dt}\right) + 2\exp\left(-\frac{9H^2}{4Dt}\right) + 2\exp\left(-\frac{25H^2}{4Dt}\right) + \cdots\right]$$

Selected values are tabulated below:

t (in min)	$c_{mid\text{-}depth}$ (in mg/L)	c_{bottom} (in mg/L)
1	4.290	0.000
2	50.509	0.022
4	105.310	4.290
6	190.260	22. 839
8	208.924	50.509
10	216.692	79.287
15	221.605	137.050
20	222.153	172.712
30	222.221	205.664
40	222.222	216.692
50	222.222	220.375
60	222.222	221.605

We note that the concentration at mid-depth increases monotonically (no intermediate maximum).

The ultimate concentration value is obtained by distributing the whole amount over the entire depth:

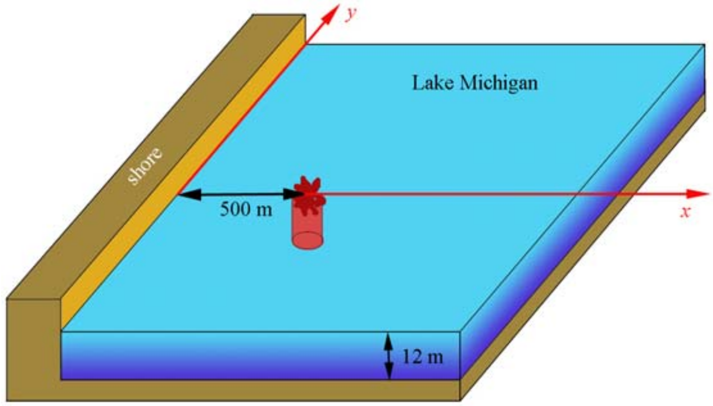
$$c_{end} = \frac{M}{A\,H} = \frac{10\,kg}{(5\,m^2)(9\,m)} = 0.222\,kg/m^3 = 0.222\,g/L = 222\,mg/L$$

The time for nearly complete mixing is given by

$$T = 0.536\,\frac{H^2}{D} = (0.536)\frac{(9\,m)^2}{(0.015\,m^2/s)} = 2894\,s = 48\min 24\,s.$$

Both results are borne out by the numerical calculations.

3. (10 points) In Lake Michigan, a barge accidentally spilled 350 kg of a conservative contaminant 500 m from shore and within a brief amount of time. Assuming rapid mixing in the vertical, a uniform depth of 12 m in the area, a straight coastline and a horizontal diffusivity of 3.0 m^2/s, trace over time the concentration of the contaminant at the location of the spill and at the nearest point along the shore. Discuss your findings.



Because of complete mixing over the vertical, spreading occurs only in both horizontal directions, and this is a case of two-dimensional diffusion. There is no mean motion and the release was brief and localized. For axes, choose *x* going offshore, starting from the coast and *y* going alongshore, as depicted in figure above.

Given the presence of a boundary (the coast), at $x = 0$, the release occurs at $x = L = 500$ m and $y = 0$, and we need to add the contribution of a mirror release at $x = -L$ and $y = 0$. The concentration distribution is given by:

$$c(x,y,t) = \frac{M}{\sqrt{4\pi Dt}\ \sqrt{4\pi Dt}}\left[\exp\left(-\frac{(x-L)^2}{4Dt}\right) + \exp\left(-\frac{(x+L)^2}{4Dt}\right)\right]\exp\left(-\frac{y^2}{4Dt}\right)$$

where M is the amount released divided by depth (the missing dimension):
$M = 350$ kg / 12 m = 29.17 kg/m.

At the location of the spill ($x = L$ and $y = 0$), the concentration evolves according to

$$c_{spill}(t) = \frac{M}{4\pi Dt}\left[1 + \exp\left(-\frac{L^2}{Dt}\right)\right]$$

and at the point on the shore nearest to the spill ($x = 0$, $y = 0$), it evolves according to

$$c_{shore}(t) = \frac{M}{2\pi Dt}\exp\left(-\frac{L^2}{4Dt}\right)$$

With $D = 3.0$ m^2/s and $L = 500$ m, sample values are:

Time t	c at spill (in mg/m^3)	c at shore (in mg/m^3)
30 min	429.8	8.08 x 10^{-3}
1 hour	214.9	1.318
2 hours	107.5	11.90
3 hours	71.67	20.82
4 hours	53.89	25.29
5 hours	43.40	27.02
5 hours 47 min	37.84	**27.32**
6 hours	36.57	27.31
8 hours	28.35	26.06
9 hours 30 min	24.60	24.60
12 hours	20.51	22.11
1 day	12.37	14.07
2 days	7.24	7.94
3 days	5.15	5.51

We note that the concentration decreases monotonically at the place of the spill, which is expected because of the spread away from the accident. At, the shore, the concentration first rises and then decays. To find exactly when the concentration reaches its maximum at the coast, we take the time derivative of $c_{shore}(t)$ and set it to zero. The time when this occurs is

$$t = \frac{L^2}{4D} = \frac{(500\ m)^2}{(4)(3.0\ m^2/s)} = 20{,}833\ s = 5\ hrs\ 47\ \text{min}$$

and the value of the maximum is obtained by substitution of this time value:

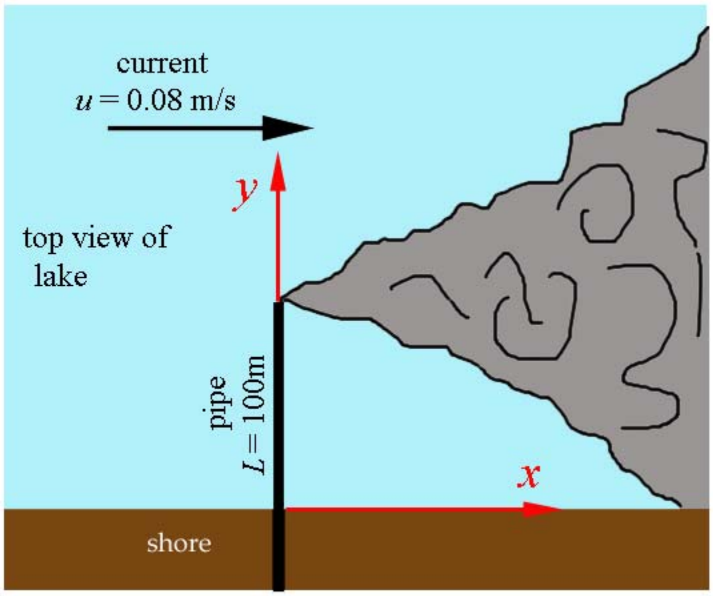
$$c_{shore-\max} = \frac{2M}{\pi L^2}\exp(-1) = 0.2342\frac{M}{L^2} = 27.32\ mg/m^3$$

We further note that after some time (9 hours and 30 minutes), the concentration is higher at the shore than at the spill. This is somewhat counterintuitive.

4. (10 points) Partially treated sewage with a BOD of 48 mg/L is continuously discharged at the rate of 3.0 m^3/s at the bottom of a 5-m lake by means of a pipe lying on the bottom and extending 100 m away from the shore, as depicted below. In that region, the lake waters move uniformly in the alongshore direction with a speed of 0.08 m/s. Once in the water, the sewage decays at the rate K = 0.24 /day and diffuses at the rate D = 3.0 m^2/s.

Assuming that the discharge does not significantly alter the water flow (no jet), that vertical mixing takes place almost instantaneously (thanks to the buoyancy of the sewage, which is slightly warmer than the lake water) and that the situation is highly advective (high Peclet number), determine the 2D horizontal distribution of sewage concentration.

If you had to write a piece for a local newspaper describing the effect that this sewage is having on the water near the shore, what would you say?



This is a problem with no diffusion in the downstream direction x (assumption of a highly advective situation) and no diffusion in the vertical direction z (assumption of rapid mixing over the depth of the lake). The only direction of diffusion is the horizontal transverse direction y (see axes in sketch above). We can assume that the discharge is constant over time, thus leading to steady state. There is advection (speed u) and decay (rate K). The equation governing the concentration of the BOD is then

$$u \frac{\partial c}{\partial x} = D \frac{\partial^2 c}{\partial y^2} - K\, c$$

Using travel time $t = x/u$ and replacing the traditional x coordinate by y in this case, we obtain the following solution by suitably amending the prototypical solution:

$$c(x,y) = \frac{M}{\sqrt{4\pi D\ x/u}} \exp\left(-\frac{y^2}{4D\ x/u} - K\frac{x}{u}\right)$$

Minding now the fact that the discharge is not at $y = 0$ but at $y = L = 100$ m away from the shore, and that the shoreline acts as an impermeable boundary necessitating an image, the preceding solution needs to be augmented to:

$$c(x,y) = \frac{M}{\sqrt{4\pi\ D\ x/u}} \exp\left(-K\frac{x}{u}\right)\left[\exp\left(-\frac{(y-L)^2}{4D\ x/u}\right) + \exp\left(-\frac{(y+L)^2}{4D\ x/u}\right)\right]$$

We are naturally interested in what happens along the shoreline. We thus set the coordinate y to zero and obtain:

$$c_{shore}(x) = c(x, y=0) = \frac{2M}{\sqrt{4\pi D\ x/u}} \exp\left(-K\frac{x}{u}\right) \exp\left(-\frac{u\ L^2}{4D\ x}\right)$$

$$= \frac{M}{\sqrt{\pi\ Dx/u}} \exp\left(-\frac{u\ L^2}{4D\ x} - \frac{K\ x}{u}\right)$$

It remains to determine the front coefficient M. This is the amount released per missing dimensions, that is,

$$M = \frac{mass\ released}{(x-length)(z-length)} = \frac{mass\ released\ /\ time}{(x-length\ /\ time)(z-length)}$$

$$= \frac{(mass\ /\ volume)(volume\ /\ time)}{(current\ speed)(depth)} = \frac{(48\ mg\ /\ L)(3.0\ m^3\ /\ s)}{(0.08\ m\ /\ s)(5\ m)} = 360\ g\ /\ m^2$$

Remaining values are:

$D = 3.0$ m^2/s
$u = 0.08$ m/s
$L = 100$ m
$K = 0.24$ /day = 2.778×10^{-6} /s

Substitution of numerical values yields

$$c_{shore}(x) = \frac{33.167}{\sqrt{x}} \exp\left(-\frac{66.67}{x} - 3.472 \times 10^{-5}\ x\right)$$

in g/m^3 with x given in meters. Sample values are:

position x (in m or km)	concentration c (in $g/m^3 = mg/L$)
10 m	0.013
50 m	1.234
75 m	1.570
100 m	1.697
150 m	**1.727**
200 m	1.669
250 m	1.593
500 m	1.276
1 km	0.948
10 km	0.233

There is a maximum somewhere along the coast. More precise calculations in the vicinity of 150 m yield**: c_{max} = 1.734 g/m^3 at distance x = 132 m**.

The latter values should be quoted in the newspaper article (maximum value and where it occurs).