

Appendix B: Wave Kinematics

(August 31, 2009) **SUMMARY:** Because numerous geophysical flow phenomena can be interpreted as waves, some understanding of basic wave properties is required in the study of geophysical fluid dynamics. The concepts of wavenumber, frequency, dispersion relation, phase speed and group velocity are introduced and given physical interpretations.

B.1 Wavenumber and wavelength

For simplicity of presentation and easier graphical representation, we will consider here a two-dimensional plane wave, namely, a physical signal occupying the (x, y) plane, evolving with time t and with straight crest lines. The prototypical wave form is the sinusoidal function, and so we assume that a physical variable of the system, denoted by a and being pressure, a velocity component or whatever, evolves according to

$$a = A \cos(k_x x + k_y y - \omega t + \phi). \quad (\text{B.1})$$

The coefficient A is the wave *amplitude* ($-A \leq a \leq +A$), whereas the argument

$$\alpha = k_x x + k_y y - \omega t + \phi \quad (\text{B.2})$$

is called the *phase*. The latter consists of terms that vary with each independent variable and a constant ϕ , called the reference phase. The coefficients k_x , k_y , and ω of x , y , and t , respectively, bear the names of *wavenumber in x* , *wavenumber in y* and *angular frequency*, most often abbreviated to *frequency*. They indicate how rapidly the wave undulates in space and how fast it oscillates in time.

Equivalent expressions for the wave signal are

$$a = A_1 \cos(k_x x + k_y y - \omega t) + A_2 \sin(k_x x + k_y y - \omega t), \quad (\text{B.3})$$

where $A_1 = A \cos \phi$ and $A_2 = -A \sin \phi$, and

$$a = \Re \left[A_c e^{i(k_x x + k_y y - \omega t)} \right], \quad (\text{B.4})$$

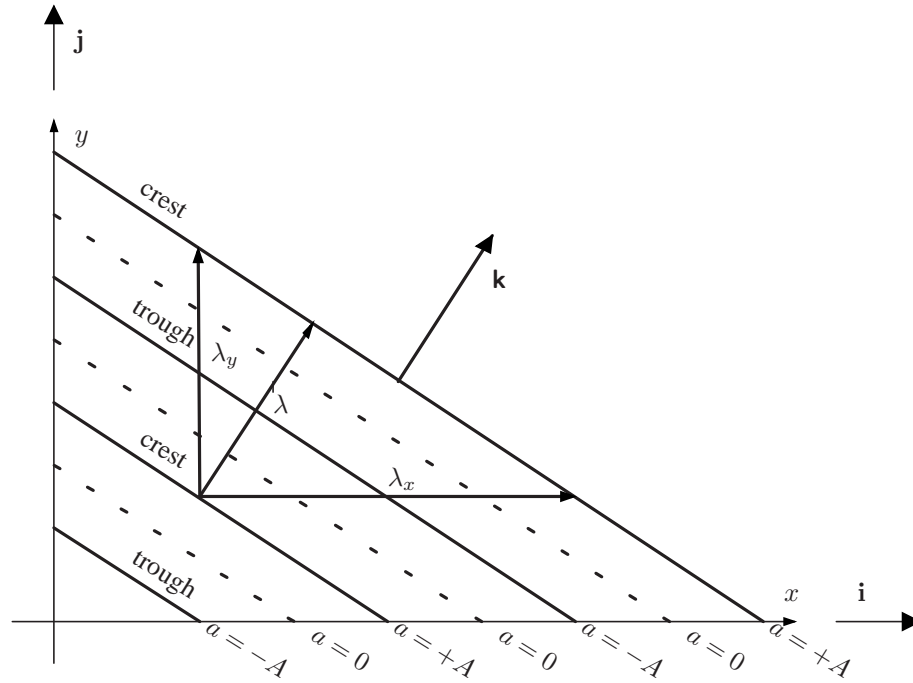


Figure B-1. Instantaneous phase lines of a plane two-dimensional wave signal. The lines are straight and parallel. The distances from crest to nearest crest along the x - and y -axes are λ_x and λ_y , respectively, whereas the wavelength λ is the shortest diagonal distance from crest line to nearest crest line.

where the notation $\Re[\]$ means “the real part of” and $A_c = A_1 - iA_2 = A e^{i\phi}$ is a complex amplitude coefficient. The choice of mathematical representation is generally dictated by the problem at hand. Formulation (B.3) is helpful in the discussion of problems exhibiting coexisting signals that are either in perfect phase or in quadrature, whereas formulation (B.4) is preferred when a given dynamical system is subjected to a wave analysis. Here, we will use formulation (B.1).

A wave *crest* is defined as the line in the (x, y) plane and at time t along which the signal is maximum ($a = +A$); similarly, a *trough* is a line along which the signal is minimum ($a = -A$). These lines and, in general, all lines along which the signal has a constant value at an instant in time are called *phase lines*. In a plane wave, as the one considered here, all crests, troughs and other phase lines are straight lines. Figure B-1 depicts a few phase lines in the case of positive wavenumbers k_x and k_y .

Because of the oscillatory behavior of the sinusoidal function, crest lines recur at constant intervals, thus giving the wavy aspect to the signal. The distance over which the signal repeats itself in the x -direction is the distance over which the phase portion $k_x x$ increases by 2π , that is,

$$\lambda_x = \frac{2\pi}{k_x}. \quad (\text{B.5})$$

Similarly, the distance over which the signal repeats itself in the y -direction is

$$\lambda_y = \frac{2\pi}{k_y}. \quad (\text{B.6})$$

The quantities λ_x and λ_y are called the wavelengths in the x - and y -directions. They are the wavelengths seen by an observer who would detect the signal only through slits aligned with the x and y axes. The actual *wavelength*, λ , of the wave is the shortest distance from the crest to nearest crest (Figure B-1) and is, therefore, smaller than either λ_x and λ_y . Elementary geometric considerations provide

$$\frac{1}{\lambda^2} = \frac{1}{\lambda_x^2} + \frac{1}{\lambda_y^2} = \frac{k_x^2 + k_y^2}{4\pi^2}$$

or

$$\lambda = \frac{2\pi}{k}, \quad (\text{B.7})$$

where k , called the *wavenumber*, is defined as

$$k = \sqrt{k_x^2 + k_y^2}. \quad (\text{B.8})$$

Note that since λ^2 is not the sum of λ_x^2 and λ_y^2 , the pair (λ_x, λ_y) does not make a vector. On the other hand, the pair (k_x, k_y) can be used to define the *wavenumber vector*

$$\mathbf{k} = k_x \mathbf{i} + k_y \mathbf{j}, \quad (\text{B.9})$$

where \mathbf{i} and \mathbf{j} are the unit vectors aligned with the axes (Figure B-1). In this fashion, the wavenumber k is the magnitude of the wavenumber vector \mathbf{k} .

By definition, phase lines at any given time correspond to lines of constant values of the expression $k_x x + k_y y = \mathbf{k} \cdot \mathbf{r}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ is the vector position. Geometrically, this means that a phase line is the locus of points whose vectors from the origin share the same projection onto the wavenumber vector. These points form a straight line perpendicular to \mathbf{k} , and therefore the wavenumber vector points perpendicularly to all phase lines (Figure B-1), that is, in the direction of the undulation.

B.2 Frequency, phase speed, and dispersion

Let us now turn our attention to the temporal evolution of the wave signal. At a fixed position (x and y given), an observer sees an oscillatory signal. The interval of time between two consecutive instants at which the signal is maximum is the time taken for the portion ωt of the phase to increase by 2π . It is called the *period*, which is

$$T = \frac{2\pi}{\omega}. \quad (\text{B.10})$$

Let us now follow a particular crest line ($a = A$) from a certain time t_1 to a later time t_2 and note the time interval $\Delta t = t_2 - t_1$. During this time interval, the wave crest has progressed from one position to another (Figure B-2). The intersection with the x -axis has

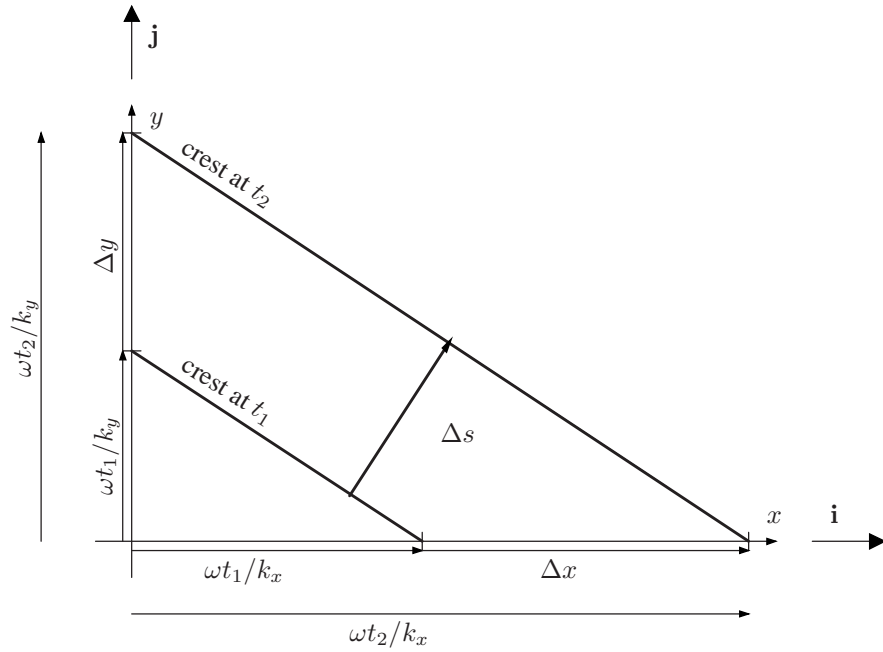


Figure B-2. Progress of a wave crest from time t_1 to time t_2 . The ratio of the distance traveled, Δs , to the time interval $\Delta t = t_2 - t_1$ is the phase speed.

translated over the distance $\Delta x = \omega t_2/k_x - \omega t_1/k_x = \omega \Delta t/k_x$ in time Δt . This defines the propagation speed of the wave along the x -direction:

$$c_x = \frac{\Delta x}{\Delta t} = \frac{\omega}{k_x}. \quad (\text{B.11})$$

Similarly, the propagation speed along the y -direction is the distance $\Delta y = \omega t_2/k_y - \omega t_1/k_y$ divided by the time interval Δt , or

$$c_y = \frac{\Delta y}{\Delta t} = \frac{\omega}{k_y}. \quad (\text{B.12})$$

But, these speeds are only speeds in particular directions. The true propagation speed of the wave is the distance Δs , measured perpendicularly to the crest line (Figure B-2), covered by this crest line during the time interval Δt . Again, elementary geometric considerations provide

$$\frac{1}{\Delta s^2} = \frac{1}{\Delta x^2} + \frac{1}{\Delta y^2},$$

from which we deduce

$$\Delta s = \frac{\omega \Delta t}{k},$$

where k is the wavenumber defined in (B.8). The propagation speed of the crest line is thus

$$c = \frac{\Delta s}{\Delta t} = \frac{\omega}{k}. \quad (\text{B.13})$$

Because all phase lines propagate at the same speed (so that the wave preserves its sinusoidal form over time), the quantity c is called the *phase speed*. Note that because c^2 is not equal to $c_x^2 + c_y^2$ (in fact, c is less than both c_x and c_y !), the pair (c_x, c_y) does not constitute a physical vector. The direction of phase propagation, as discussed before, is parallel to the wavenumber vector \mathbf{k} .

In general, the expression (B.1) of the wave signal appears as the solution to a particular dynamical system. Therefore, it must somehow be constrained by the physics of the problem, and not all its parameters can be varied independently. Let us suppose that the system under consideration is initially unchanging in time (state of rest or steady flow) and that at time $t = 0$, it is perturbed spatially according to a sinusoidal distribution of wavenumbers k_x and k_y in the x - and y -directions, respectively, and of amplitude A for the variable a . Intuition leads us to anticipate that subsequent to this perturbation, the system will react in a time-dependent fashion. If this reaction takes the form of a wave, it will have a frequency ω determined by the system. Therefore, the frequency can be viewed as dependent upon the wavenumber components k_x and k_y and the amplitude A . In most instances, the system's response is a wave because the set of equations representing the physics is linear, and when this is the case, the mathematical analysis yields a frequency that is independent of the amplitude of the perturbation. Therefore, ω is typically a function of k_x and k_y only.

If the frequency is a function of the wavenumber components, so is the phase speed:

$$c = \frac{\omega(k_x, k_y)}{\sqrt{k_x^2 + k_y^2}} = c(k_x, k_y).$$

Physically, this implies that the various waves of a composite signal will all travel at different speeds, causing a distortion of the signal over time. In particular, a localized burst of activity, which by virtue of the Fourier-decomposition theorem contains waves of many different wavelengths, will be progressively less localized as time goes on. This phenomenon is called *dispersion*, and the mathematical function that relates the frequency ω to the wavenumber components k_x and k_y bears the name of *dispersion relation*.

The dispersion relation can be represented, in two dimensions, as a set of curves in the (k_x, k_y) plane along which ω is a constant. Figure B-3 provides an example. At one dimension ($k_x = k, k_y = 0$) or at two dimensions when the physical system is isotropic (ω function of k only), a single ω -versus- k curve suffices to represent the dispersion relation.

In some special physical systems, the dispersion relation reduces to a single proportionality between frequency ω and wavenumber k . The phase speed is then the same for all wavenumbers, all waves travel in perfect accord, and the total signal retains its shape as time evolves. Such a wave is called a *nondispersive* wave.

B.3 Group velocity and energy propagation

In general, a wave pattern consists of more than a single wave. A series of waves are superimposed, leading to constructive and destructive interference. In areas where the waves are

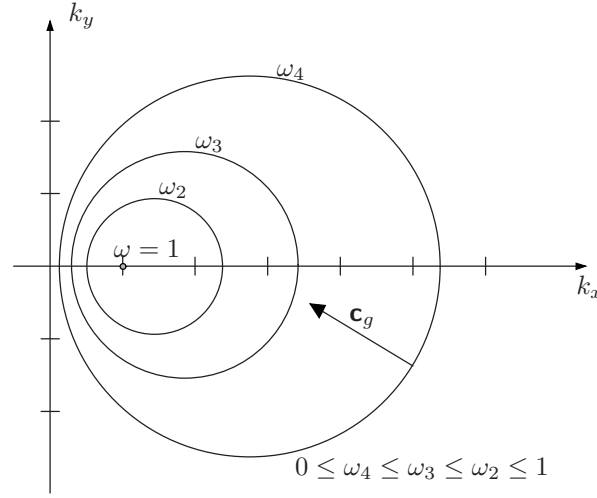


Figure B-3. Graphic representation of the dispersion relation $\omega = 2k_x/(k_x^2 + k_y^2 + 1)$ by curves of constant ω values in the (k_x, k_y) wavenumber plane.

interfering constructively, the wave amplitude is greater and the energy level is higher than in areas where the waves interfere destructively. Therefore, energy distribution is a property of a set of waves rather than of a single wave. (It can be said that a single wave has a uniform energy distribution.) Energy propagation by a set of waves depends on how interference patterns move about and is generally not the average speed of the waves present. To illustrate the principles and determine the speed at which energy propagates, let us restrict our attention to two one-dimensional waves and, more precisely, to two waves of equal amplitude and nearly equal wavenumbers:

$$a = A \cos(k_1 x - \omega_1 t) + A \cos(k_2 x - \omega_2 t), \quad (\text{B.14})$$

where the wavenumbers k_1 and k_2 are close to their average $k = (k_1 + k_2)/2$, and the difference $\Delta k = k_1 - k_2$ is much smaller ($|\Delta k| \ll |k|$). Because both waves obey the single dispersion relation of the dynamical system, $\omega = \omega(k)$, the two frequencies $\omega_1 = \omega(k_1)$ and $\omega_2 = \omega(k_2)$ are close to their average $\omega = (\omega_1 + \omega_2)/2$, which is much larger than their difference $\Delta\omega = \omega_1 - \omega_2$ ($|\Delta\omega| \ll |\omega|$). In expression (B.14), the two reference phases were set to zero, which can always be done under suitable choices of space and time origins.

A trigonometric manipulation transforms expression (B.14) into

$$a = 2A \cos\left(\frac{\Delta k}{2} x - \frac{\Delta\omega}{2} t\right) \cos(kx - \omega t), \quad (\text{B.15})$$

which now appears as the product of two waves. The second cosine function represents an average wave, of wavenumber and frequency between those of the two individual waves comprising the signal. The first cosine function, on the other hand, involves a much smaller wavenumber (i.e. much longer wavelength) and a much lower frequency. Over the cycle of the shorter (k, ω) wave, the longer wave appears almost unchanging. In other words, the $(k,$

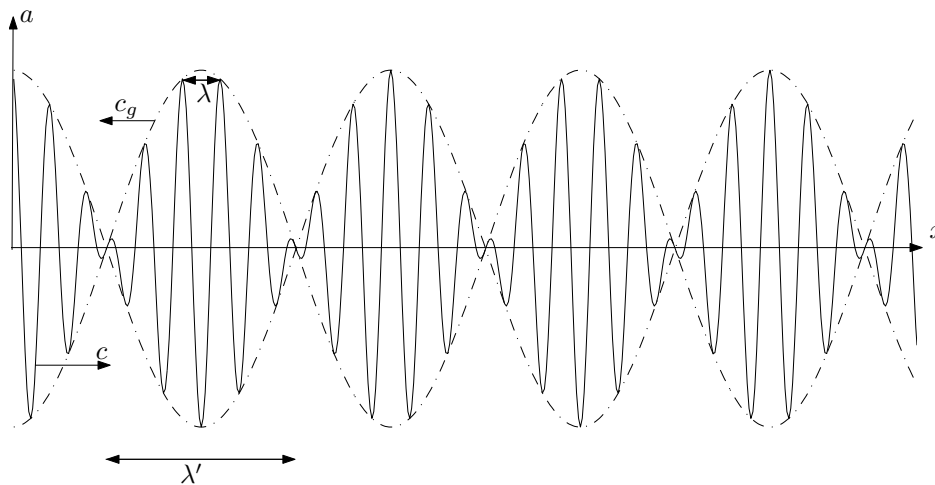


Figure B-4. The interference pattern of two one-dimensional waves with close wavenumbers. While the wave crests and troughs propagate at the speed $c = \omega/k$, the envelope (dashed line) propagates at the group velocity $c_g = d\omega/dk$.

ω) wave appears modulated; its amplitude, $2A \cos[(\Delta kx - \Delta\omega t)/2]$, is slowly varying in space and time, as seen in Figure B-4.

Although the wave signal exhibits a wavelength from crest to trough to the next crest equal to $\lambda = 2\pi/k$, the envelope has a much longer wavelength, $\lambda' = \frac{1}{2} [2\pi/(\Delta k/2)] = 2\pi/\Delta k$. The wave pattern is a succession of wave bursts, each of length λ' . Within each burst, the wave propagates at the phase speed $c = \omega/k$, while the burst travels at the speed $c' = \Delta\omega/\Delta k$.

Considering an infinitesimal wavenumber difference, we are led to define

$$c_g = \frac{d\omega}{dk}. \quad (\text{B.16})$$

Because this is the propagation speed of a burst, or group of similar waves, it is called the *group velocity*. Energy is associated with each group, and so the group velocity is also the velocity at which energy is carried by the waves.

The preceding wave description relies on the existence of two waves of identical amplitudes. When two waves do not have equal amplitude, say A_1 and A_2 , destructive interference is nowhere complete (the weak wave cannot completely cancel the strong wave), and there is no clear pinch-off between wave bursts. Rather, the modulating envelope undulates between the values $A_1 + A_2$ and $|A_1 - A_2|$ on the positive side and $-(A_1 + A_2)$ and $-|A_1 - A_2|$ on the negative side. It remains, however, that regions of constructive interference, and thus of higher energy level, propagate at the group velocity.

The theory can easily be extended to multi-dimensional waves. At two dimensions, for example, we define the group velocities in the x - and y -directions, respectively, as

$$c_{gx} = \frac{\partial \omega}{\partial k_x}, \quad c_{gy} = \frac{\partial \omega}{\partial k_y}, \quad (\text{B.17})$$

given the dispersion relation $\omega(k_x, k_y)$. Because these expressions are the components of the gradient of the function ω in the (k_x, k_y) wavenumber space, they can be interpreted as the components of a physical vector depicting the group velocity

$$\mathbf{c}_g = \nabla_k \omega, \quad (\text{B.18})$$

where ∇_k stands for the gradient operator with respect to the variables k_x and k_y . On the two-dimensional diagram (Figure B-3), this vector group velocity points perpendicularly to the ω curves, toward the higher values of ω . Aligning the k_x - and k_y -axes with the x - and y -axes of the plane provides the direction of energy propagation in space.

Generalization to the three-dimensional space is immediate. An example is the internal wave discussed extensively in Chapter 13.

Analytical Problems

- B-1.** In waters deeper than half the wavelength, surface gravity waves obey the dispersion relation $\omega = \sqrt{gk}$, where g is the gravitational acceleration ($g = 9.81 \text{ m/s}^2$). For these waves, show that the wavelength is proportional to the square of the period. At which speed does a 10 m-long wave travel?
- B-2.** Show that the group velocity of deep-water waves (see Problem B-1) is always less than the phase speed.
- B-3.** A former sea captain recounts a stormy night in the middle of the North Atlantic when he observed waves with wavelengths of a few meters passing his 51-m-long ship in less than 3 s. Should you believe him?
- B-4.** Suppose that you are in the middle of the ocean and off in the distance you see a storm. A little while later, you observe the passage of surface gravity waves of wavelength 5 m. Two hours later, you still observe gravity waves, but now their wavelength is 2 m. How far away was the storm?
- B-5.** Find the frequency ω of a Kelvin wave of wavenumber k (Section 9.2). Is the Kelvin wave dispersive?
- B-6.** Show that for inertia-gravity waves [$\omega^2 = f^2 + gH(k_x^2 + k_y^2)$; Section 9.3], the group velocity is always less than the phase speed. In which limit does the group velocity approach the phase speed?
- B-7.** Demonstrate that when the frequency ω is a function of the ratio k_x/k_y , the energy propagates perpendicularly to the phase.

B-8. Given the dispersion relation of internal waves in a vertical plane (see Section 13.2),

$$\omega = N \frac{k_x}{\sqrt{k_x^2 + k_z^2}},$$

where N is a constant, k_x is the horizontal wavenumber and k_z is the vertical wavenumber, show that phase and energy always propagate in the same horizontal direction but in opposite vertical directions.

Numerical Exercises

B-1. Using animated graphics, display a time sequence ($t = 0$ to 10π by steps of $\pi/4$) of the double wave

$$a(x, t) = A_1 \cos(k_1 x - \omega_1 t) + A_2 \cos(k_2 x - \omega_2 t)$$

with $A_1 = A_2 = 1$, $k_1 = 1.9$, $k_2 = 2.1$, $\omega_1 = 2.1$, $\omega_2 = 1.9$, and for x ranging from 0 to 100. A suggested step in x is 0.25. Notice how the short waves [of wavelength = $4\pi/(k_1 + k_2) = \pi$] travel toward increasing x at the speed $c = (\omega_1 + \omega_2)/(k_1 + k_2) = +1$, while the wave envelope [of wavelength = $2\pi/(k_2 - k_1) = 10\pi$] travels in the opposite direction at speed $c_g = (\omega_1 - \omega_2)/(k_1 - k_2) = -1$. This unequivocally demonstrates the nonintuitive fact that the energy propagation may well propagate in the direction opposite to the advancing crests and troughs. In other words, it is not impossible for energy to be transported up-wave.

Variations of this exercise can include uneven amplitudes (e.g., $A_1 = 1$ and $A_2 = 0.5$) and modified values for the wavenumbers and frequencies.

B-2. Using animated graphics, use the same function as in exercise B-1 with $k_1 = 0.35$, $k_2 = 0.5$, $\omega_1 = 0.5$, and $\omega_2 = 0.35$ the other values unchanged. Show the evolution of a and then of $a^2/2$. Can you explain the apparently shorter waves?

B-3. Given a dispersion relation

$$\omega = \frac{k}{(k^2 + 1)}$$

analyze now the signal composed of two waves

$$a(x, t) = A_1 \cos(k_1 x - \omega(k_1)t) + A_2 \cos(k_2 x - \omega(k_2)t),$$

where ω is calculated using the dispersion relation. As before, show the evolution for $A_1 = A_2 = 1$ in the following situations

- $k_1 = k_2 = 0.5$,

- $k_1 = k_2 = 2$
- $k_1 = 1.95, k_2 = 2.05$
- $k_1 = 0.45, k_2 = 0.55$

Can you explain the behaviour? (*Hint*: Plot the dispersion relation.)

B-4. Redo exercise B-1 with $k_1 = 1$, $A_1 = 1$, $A_2 = 0$ and $\omega_1 = 1$. Then change the step in x to $\pi/4$ and $\pi/2$. Finally when using a step of $4\pi/3$, what do you observe?