Appendix A: Elements of Fluid Mechanics

(August 31, 2009) **SUMMARY**: Basic principles of continuum mechanics are presented and particularized for fluid flows. It is shown how bugets can be established on infinitesimal volumes, relying on the hypothesis of continuity of matter. Because sometimes spherical or cylindrical coordinates are better suited for a given problem, the basic operators expressed in those coordinate systems are provided. Finally the link between vorticity and rotation is outlined.

A.1 Budgets

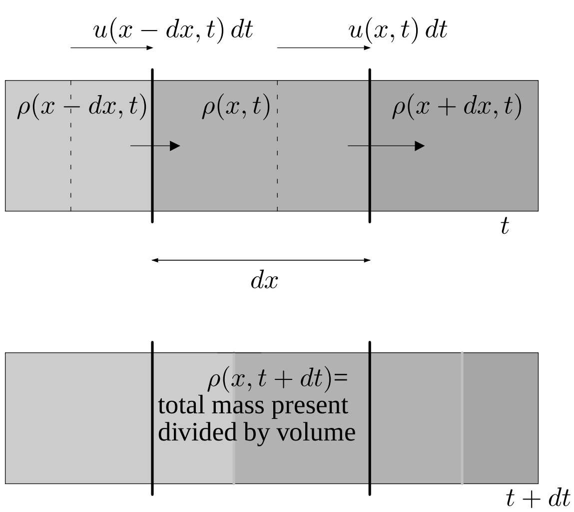

Figure A-1 One-dimensional mass conservation.

The simplest budget calculation uses the fact that mass is conserved. In this case a one-dimensional budget over an infinitesimal segment of Figure A-1 simply states that the mass within the segment at one moment is the mass at a previous moment augmented by the net inflow of mass:

$$\underbrace{\rho(x,t+dt)\,dx}_{\text{mass at time } t+dt} = \underbrace{\rho(x,t)\,dx}_{\text{mass at time } t} + \underbrace{u(x-dx,t)dt\,\rho(x-dx,t)}_{\text{mass entering}} - \underbrace{u(x,t)dt\,\rho(x,t)}_{\text{mass exiting}} \tag{A.1}$$

Dividing by time interval dt and space interval dx this budget yields

$$\frac{\rho(x,t+dt)-\rho(x,t)}{dt} + \frac{u(x,t)\rho(x,t)-u(x-dx,t)\rho(x-dx,t)}{dx} = 0 \tag{A.2}$$

from which, for infinitesimally small increments we find the one-dimensional mass conservation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0. \tag{A.3}$$

Note that we assumed such an infinitesimal limit exists, meaning that "infinitesimal" is extremely small compared to the scales of macroscopic properties, yet large compared to the size of the molecules constituting the material for which the budget is calculated. This is the essence of a continuum mechanics approach.

Similarly, in a three-dimensional domain (Figure A-2), the budget calculation yields:

$$\begin{aligned}
\rho(x,,y,z,t+dt)\,dx\,dy\,dz &= \rho(x,y,z,t)\,dx\,dy\,dz \\
+u(x-dx,y,z,t)dt\,dy\,dz\,\rho(x-dx,y,z,t) &- u(x,y,z,t)dt\,dy\,dz\,\rho(x,y,z,t) \\
+v(x,y-dy,z,t)dt\,dx\,dz\,\rho(x,y-dy,z,t) &- v(x,y,z,t)dt\,dx\,dz\,\rho(x,y,z,t) \\
+w(x,y,z-dz,t)dt\,dx\,dy\,\rho(x,y,z-dz,t) &- w(x,y,z,t)dt\,dx\,dy\,\rho(x,y,z,t)
\end{aligned}$$

From there, dividing by the infinitesimal volume $dx\,dy\,dz$ and time difference dt we get

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0, \tag{A.4}$$

the mass conservation equation.

Instead of making a budget on mass, we can also perform budgets on energy and momentum. We illustrate the approach on momentum in a two-dimensional case. Hence ρu is the momentum (per unit of volume) which can be changed by forces acting on the parcel (Newton's law) and by in- and outflow of momentum. The budget of Figure A-3 states that in two dimensions, Newton's law requests a change in momentum (per unit volume) which is due to the sum of applied forces (per unit volume) and the budget of momentum fluxes. The latter fluxes can be calculated similarly to the mass fluxes of (A.1), except that ρ is now to be replaced by momentum.

The forces applied to the box are

$$\begin{aligned}
\rho(x,y,t)f_x(x,y,t)\ +p(x-dx,y,t)\,dy - p(x,y,t)\,dy &+ \tau^{xy}(x,y)\,dx - \tau^{xy}(x,y-dy)\,dx \\
&+ \tau^{xx}(x,y)\,dy - \tau^{xx}(x-dx,y)\,dy
\end{aligned}$$

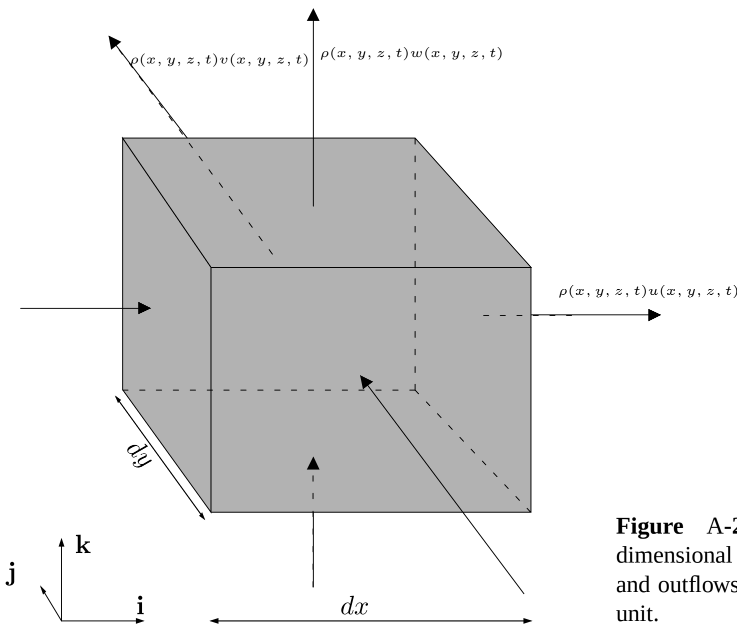

Figure A-2 Infinitesimal three-dimensional volume with mass inflow and outflows per time unit and surface unit.

where the stresses τ depend on the nature of the matter. Note that $\tau^{xy}(x, y)$ and $\tau^{yx}(x, y)$ are stresses in different directions (Figure A-3) but with the same amplitude. This equality of stresses τ^{xy} and τ^{yx} can be understood as follows: if these stresses had not the same intensity, the infinitesimal volume would be subjected to a torque, which could not be balanced by an internal torque, hence the need for $\tau^{xy} = \tau^{yx}$. A common constitutive equation relates the stress to the shear (see Figure A-4). In any case, the momentum budget is controlled by the stresses, irrespectively of their formulation. With these forces and the in- and outflow of momentum, dividing by $dx\, dy$, we get the momentum budget

$$\begin{aligned} \frac{\partial}{\partial t}(\rho u) \quad &+ \quad \frac{\partial}{\partial x}(\rho u u) + \frac{\partial}{\partial y}(\rho u v) + \\ &= \quad \rho f_x - \frac{\partial p}{\partial x} + \frac{\partial \tau^{xy}}{\partial y} + \frac{\partial \tau^{xx}}{\partial x} \end{aligned} \tag{A.5}$$

and similarly for y direction. Generalization to the three-dimensional case is straightforward and leads to (3.2) with (3.3).

Note that generally Newton's law is presented by following a given mass (called a Lagrangian approach), rather than by performing a budget over a fixed part of space (called an Eulerian approach). Since the physical law is the same, we should be able to reach the same governing equations by both approaches. To show that this is possible, we define the Eulerian derivative of a field $F(x, y, z, t)$, which is a property of the flow field as

$$\frac{\partial F}{\partial t} = \text{derivative of } F \text{ with respect to } t, \text{ at fixed } x, y, z \tag{A.6}$$

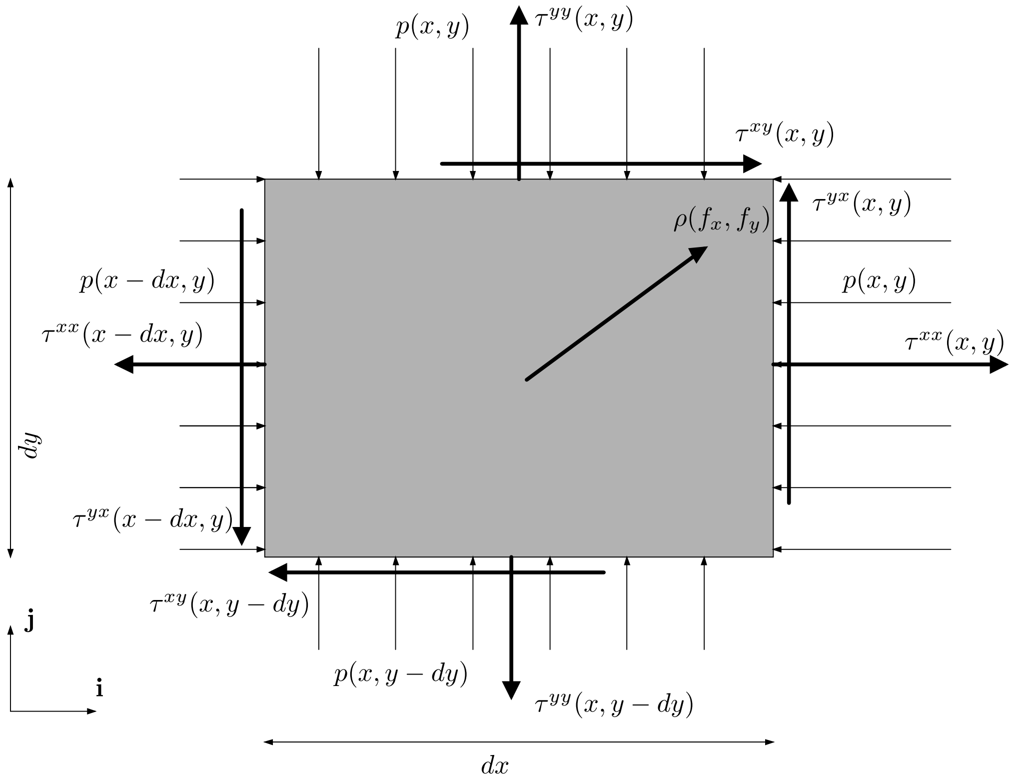

Figure A-3 Two-dimensional situation with forces acting on the fluid parcel. Note that surface stresses have a positive value in opposite directions on facing sides. This is due to the action-reaction between infinitesimal volumes. Also note how pressure acts in the normal direction to the faces. Force (ρf_x , ρf_y) acts as a body force.

Hence, this is the time change of F we see as a fixed observer.

On the contrary, in a Lagrangian approach we look at the changes moving with a fluid parcel, whose position changes over time according to $(x, y, z) = (x(t), y(t), z(t))$. This time dependence of the coordinates describes the trajectory of the fluid parcel. Now the time change of F, taking the displacement into account, is the time change of F we observe when looking at the same fluid parcel over time

$$\frac{dF}{dt} = \text{derivative of } F \text{ with respect to } t\text{, moving according to } (x, y, z) = (x(t), y(t), z(t)) \tag{A.7}$$

This change of F for a fluid parcel can be calculated mathematically by the chain-rule of derivatives:

$$\frac{dF}{dt} = \frac{dx}{dt}\frac{\partial F}{\partial x} + \frac{dy}{dt}\frac{\partial F}{\partial y} + \frac{dz}{dt}\frac{\partial F}{\partial z} + \frac{\partial F}{\partial t} \tag{A.8}$$

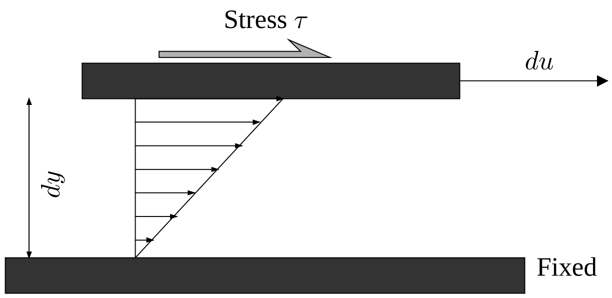

Figure A-**4** Creeping flow, where stress is proportional to shear $\tau \propto du/dy$.

Because $(x, y, z) = (x(t), y(t), z(t))$ is the trajectory of the fluid parcel, the change in position per time unit dx/dt is noting else than the parcel velocity u. Similarly $dy/dt = v$ and $dz/dt = w$ and we can express the Lagrangian derivative dF/dt, also called a material derivative as

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} . \tag{A.9}$$

This relates the Lagrangian derivative to the Eulerian derivative and we can switch from one approach to the other by expanding the material derivative into a local time derivative augmented by the advection contribution. This also leads to a direct interpretation of the mathematical rule (A.9). In a one-dimensional case (Figure A-5) the change following a parcel is indeed the local change plus the change due to the lateral movement (advection).

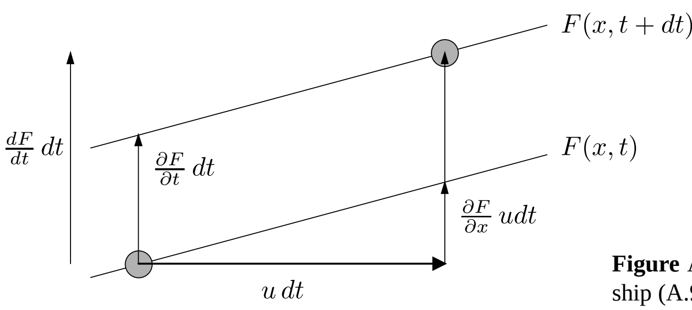

Figure A-5 Interpretation of relationship (A.9)

The Lagrangian and Eulerian formulations also allow to manipulate mass-conservation formulation (A.4) by using the material derivative (A.9) to express mass conservation as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{d\rho}{dt} = \frac{1}{\mathfrak{v}}\frac{d\mathfrak{v}}{dt} \tag{A.10}$$

with $\mathfrak{v} = 1/\rho$ being the volume taken by a given mass. Hence a positive divergence means that this volume is dilated and measures the relative change of this volume. Similarly, in two-dimensions, the divergence measures the relative change in a surface element $\mathcal{S}$ containing a given mass:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{1}{\mathcal{S}}\frac{d\mathcal{S}}{dt} \tag{A.11}$$

It is left as an exercise to the reader to formulate the momentum equation in a Lagrangian way and to interpret the resulting equation.

A.2 Equations in spherical coordinates

The preceding equations assume a Cartesian system of coordinates and thus hold only if the dimension of the domain under consideration is much shorter than the earth's radius. Should the domain dimensions be comparable to the size of the planet, the x–, y– and z–axes need to be replaced by spherical coordinates, and curvature terms enter all equations. Equations (3.1) through (3.3) become:

JMB from ⇓
JMB to ⇑

$$\frac{\partial}{\partial t}(\rho\cos\varphi) + \frac{\partial}{\partial\lambda}\left(\frac{\rho u}{r}\right) + \frac{\partial}{\partial\varphi}\left(\frac{\rho v\cos\varphi}{r}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\rho w\cos\varphi\right) = 0 \quad \text{(A.12a)}$$

$$\rho\left(\frac{du}{dt} - \frac{uv\,\tan\varphi}{r} + \frac{uw}{r} + f_* w - fv\right) = -\frac{1}{r\cos\varphi}\frac{\partial p}{\partial\lambda} + F_\lambda \quad \text{(A.12b)}$$

$$\rho\left(\frac{dv}{dt} + \frac{u^2\,\tan\varphi}{r} + \frac{vw}{r} + fu\right) = -\frac{1}{r}\frac{\partial p}{\partial\varphi} + F_\varphi \quad \text{(A.12c)}$$

$$\rho\left(\frac{dw}{dt} - \frac{u^2+v^2}{r} - f_* u\right) = -\frac{\partial p}{\partial r} - \rho g + F_r \quad \text{(A.12d)}$$

where φ is the latitude[8], λ the longitude and r the distance from the center of the earth (or planet or star). The components F_λ, F_φ, and F_r of the frictional force have complicated expressions and need not be reproduced here. The material derivative becomes

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u}{r\cos\varphi}\frac{\partial}{\partial\lambda} + \frac{v}{r}\frac{\partial}{\partial\varphi} + w\frac{\partial}{\partial r}. \quad \text{(A.13)}$$

For a detailed development of these equations, the reader is referred to Chapter 4 of the book by Gill (1982).

A.3 Equations in cylindrical coordinates

We can formulate the governing equations also in cylindrical coordinates with the material derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial\theta} + w\frac{\partial}{\partial z}. \quad \text{(A.14)}$$

In this case, u is the radial velocity and v the azimuthal component (positive for a parcel moving with increasing θ). Mass conservation and horizontal components of the momentum

[8]Contrary to classical spherical coordinates, we do not use the polar angle but latitude.

equations are:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r u) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0, \tag{A.15a}$$

$$\rho\left(\frac{du}{dt} - \frac{v^2}{r} + f_* w - f v\right) = -\frac{\partial p}{\partial r} + F_r\,, \tag{A.15b}$$

$$\rho\left(\frac{dv}{dt} + \frac{uv}{r} + f u\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + F_\theta\,, \tag{A.15c}$$

$$\rho\left(\frac{dw}{dt} - f_* u\right) = -\frac{\partial p}{\partial z} - \rho g + F_z \tag{A.15d}$$

where F_r, F_θ and F_z are the friction terms. The Laplacian of a scalar field ψ reads

$$\nabla^2 \psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2}\,. \tag{A.16}$$

For polar coordinates, z dependence can simply be dropped.

A.4 Vorticity and rotation

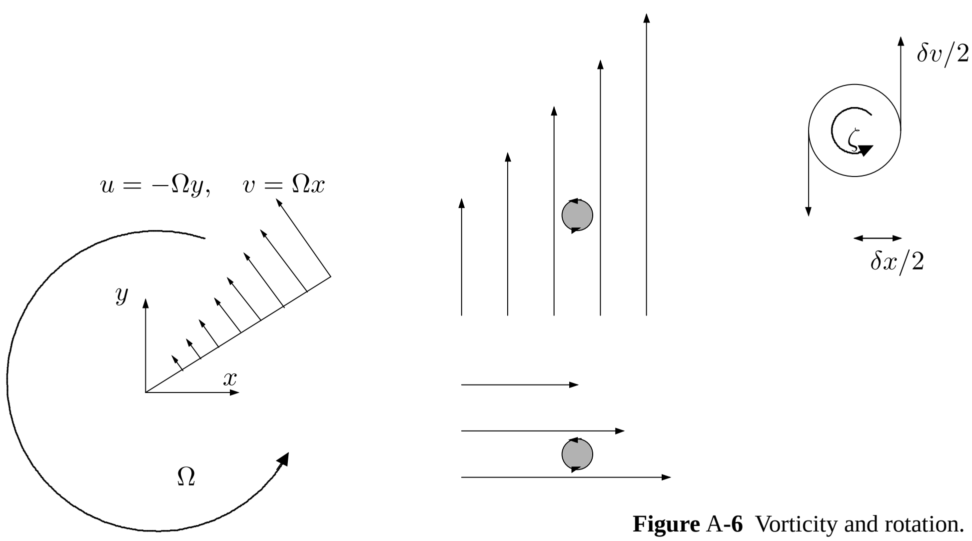

Figure A-**6** Vorticity and rotation.

Vorticity, as it name indicates, quantifies the rotation rate of a fluid. Since rotation is also defined by an axis around which the system spins, vorticity ought to be a vector. For simplicity, we will consider in detail only the case of a flow in the horizontal plane, so that rotations take place around the vertical axis. The vorticity vector is directed along this axis of rotation and its intensity is defined by

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\,. \tag{A.17}$$

First let us consider a flow in solid rotation around the origin of the axes (left part of Figure A-6). With the prescribed flow field $u = -\Omega y, \quad v = +\Omega x$, the vorticity defined by (A.17) is $\zeta = 2\Omega$, twice the rotation rate of the flow, and except for the factor 2, this seems intuitive. If the rotation is due to earth's rotation, locally with a value of $\Omega \sin \varphi$ around the vertical axis at latitude φ (see Chapter 2), the associated vorticity is $f = 2\Omega \sin \varphi$ and we understand why the Coriolis parameter f is sometimes referred to as *planetary vorticity*.

Yet a flow rotation is not the only way to create vorticity. Simple sheared flows as those depicted in the middle of Figure A-6 also yield a vorticity. For a sheared flow $v(x)$, a fluid parcel located in x is advected with velocity $v(x)$. At a distance $+\delta x/2$, velocity is higher by a value of $\delta v/2$ and in $-\delta x/2$ lower by $\delta v/2$. Hence, moving with the parcel, we see it as being sandwiched between flows of opposite velocities, leading to a rotation rate of the parcel given by

$$\zeta \frac{\delta x}{2} = \frac{\delta v}{2}. \tag{A.18}$$

For an infinitesimally small parcel, its vorticity is then $\zeta = \partial v/\partial x$, coherent with definition (A.17). Vorticity measures thus the rotation rate of a fluid parcel rather than the flow field itself. Also the sign $-$ in front of $\partial u/\partial y$ is now easily understood, as rotation is counted positive in the anticlockwise direction, which requires a shear as shown in the lower-middle part of Figure A-6.

For the three-dimensional generalization, vorticity is a vector whose components $(\zeta_x, \zeta_y, \zeta_z)$ measure the rotation rate around each axis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$:

$$\zeta_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \tag{A.19}$$

$$\zeta_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \tag{A.20}$$

$$\zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{A.21}$$

Analytical Problems

A-1. Verify that the velocity components in cylindrical coordinates are

$$u = \frac{dr}{dt}, \quad v = r\frac{d\theta}{dt}, \quad w = \frac{dz}{dt}. \tag{A.22}$$

Can you interpret these formulas? (*Hint:* Look at the definition of the material derivative for further insight.)

A-2. Calculate the vorticity of an eddy whose velocity field is given by

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x} \tag{A.23}$$

where the streamfunction ψ is defined by

$$\psi = \omega_0 L^2 \exp\left(\frac{x^2 + y^2}{L^2}\right) \tag{A.24}$$

In particular, calculate the value at the origin and in $x = 3L, y = 0$.

A-3. Assume a two-dimensional flow, for which, in cylindrical coordinates, the radial velocity component is zero, whereas the azimuthal component is only depending on r:

$$v = v(r) \; . \tag{A.25}$$

Calculate the circulation around a circle of radius R, centered around the origin and relate it to the vorticity distribution within the surface delimited by the circle. (Hint: Show that vorticity is $\zeta_z = \frac{1}{r}\,\frac{d}{dr}(rv)$.)

A-4. Knowing that the divergence of a flow is the relative change of density over time, can you derive the expression of the divergence operator in spherical and cylindrical coordinates? (*Hint:* Look at the mass-conservation equation.)

Numerical Exercises

A-1. Plot the velocity field and vorticity field from Analytical Problem A-2.