Chapter 7

Convection

SUMMARY: When temperature differences across the system create a gravitationally unstable stratification (top-heavy fluid), convection sets in. Two kinds of convection are distinguished here: top-bottom convection and penetrative convection. The chapter closes with remarks on the effects of rotation and on methods to simulate convection in computer models.

7.1 Gravitational Instability

Thermal expansion causes warmer fluid to expand thereby decreasing its density. Fluid warmer than its surrounding experiences an upward buoyancy force and tends to rise, whereas fluid colder than its surrounding tends to sink. Most often, such density differences create a stable stratification, with the lighter fluid floating atop the denser fluid. The system then allows internal waves (see Section 4.2) or limited instabilities (see Sections 5.1 and 5.2). There exist, however, cases when the vertical density gradient is inverted, due to cooling at the top of the system or heating at the bottom. The primary two examples are winter cooling of lakes (heavier fluid created near the top) and daytime heating of the lower atmosphere (lighter fluid created near the bottom[1]).

In such case, the fluid still seeks gravitational stability, with the lowest possible level of potential energy, but as long as the destabilizing heat flux persists at one of the boundaries, such ultimate stage cannot be reached, and vertical motions persist. Through these ascending and descending motions of light and heavy fluid, respectively, the system conveys heat upward. This is convection.

[1]Although the sun is above the atmosphere, its radiation occurs mostly in the visible spectrum to which the atmosphere is transparent. The opaque ground absorbs the solar radiation and re-emits it in the form of longer, infrared radiation, which effectively heats the atmosphere from below.

7.2 Rayleigh–Bénard Convection

When a thin layer of fluid is heated from below or cooled from above, the upward heat transfer can be achieved by conduction, that is, in the absence of motion on the part of the fluid because its viscosity cannot be overcome by the buoyancy forces. However, this can occur only in the rather extreme case of a very thin and very viscous fluid. Lord Rayleigh[2] studied this problem and obtained a straightforward criterion. For a horizontal fluid layer of thickness H in contact with a lower temperature T along its top surface and with a higher temperature $T + \Delta T$ along its bottom (Figure 7.1), the threshold separating the quiet from the convective regime is expressed in terms of the *Rayleigh number*:

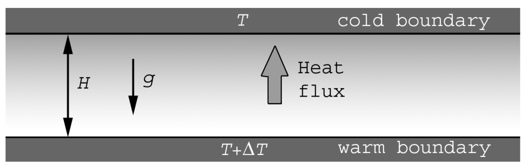

Figure 7.1: The convection problem studied by Lord Rayleigh and simulated in the laboratory by Henri Bénard.

$$Ra \;=\; \frac{\alpha g \Delta T H^3}{\nu\kappa} \,, \tag{7.1}$$

in which α is the thermal expansion coefficient, ν the kinematic viscosity, and κ is the thermal diffusivity. Values for water and air at ambient temperatures and pressures are tabulated below.

Table 7.1: Values of physical properties of fresh water (at 10°C) and air (at 15°C) at atmospheric pressure.

Physical property	Notation	Water	Air
Thermal expansion coefficient	α	2.6×10^{-4} /°C	3.5×10^{-3} /°C
Kinematic viscosity	ν	1.3×10^{-6} m²/s	1.5×10^{-5} m²/s
Thermal diffusivity	κ	1.4×10^{-7} m²/s	2.2×10^{-5} m²/s

No convective motion occurs at low values of the Rayleigh number, $Ra < 1708$, and the fluid transports heat exclusively by molecular heat diffusion. At Rayleigh numbers slightly exceeding the critical value of 1708, convection occurs in alternating patterns of upward and downward motion. Under certain conditions, a regular honeycomb pattern of hexagonal cells can be observed (Figure 7.2). This is called *Rayleigh-Bénard convection*[3]. When the Rayleigh number is several times the criti-

[2] John W. Strutt, Lord Rayleigh (1842–1919), famous English mathematician and experimental physicist

[3] in honor of Henri Bénard (1874–1939), French physicist

cal value, these patterns become unstable, oscillatory patterns arise and, for larger Ra values, convection becomes chaotic.

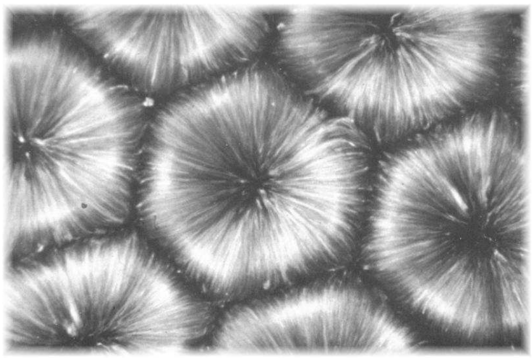

Figure 7.2: Time-lapse photograph of hexagonal Rayleigh-Bénard convective cells. Flow lines are made manifest by aluminum flakes and 10-second exposure. The fluid is rising in the center, spreading outward along the streaks, and descending at the edges of the cells. (Photo credit: M. Van Dyke, 1982, *An Album of Fluid Motion*)

7.3 Top-to-Bottom Turbulent Convection

Typical Rayleigh numbers in the atmosphere and natural bodies of water are far larger than a few thousands. For example, in a 10-m deep lake subject to a 5°C temperature difference, $Ra = 7 \times 10^{13}$ and in a 500-m thick atmospheric boundary layer subject to a 3°C temperature difference, $Ra = 4 \times 10^{16}$. Under such conditions, convection is highly turbulent and proceeds by the successive formation and release of *thermals* from the boundary subject to the destabilizing heat flux (surface for a lake, ground for the atmospheric boundary layer). Figure 7.3 shows a snapshot of thermals in a laboratory experiment conducted with water heated from below (Sparrow *et al.*, 1970).

To a given heat flux Q (in watts/m^2) and layer thickness H corresponds a vertical velocity scale that characterizes the motion of thermals. To obtain this scale, we reason that a thermal of volume V has a mass $m_{\text{thermal}} = \rho_{\text{thermal}} V$ but displaces a mass $m = \rho_{\text{ambient}} V$ of ambient fluid, subjecting it to a net weight (actual weight − buoyancy force) of $(m_{\text{thermal}} - m)g = (\rho_{\text{thermal}} - \rho_{\text{ambient}})gV$. By Newton's second law, this causes to it accelerate, from rest at the boundary where it starts, and to acquire by the time τ it reaches the other end of the domain a vertical velocity w_{thermal} equal to $(m_{\text{thermal}} - m)g\tau/m_{\text{thermal}}$, or $(\rho_{\text{thermal}} - \rho_{\text{ambient}})g\tau/\rho_{\text{thermal}}$. The travel time τ is the time it takes the thermal to cover the height H of the domain at a speed steadily increasing from zero to w_{thermal}, *i.e.*, at a mean speed $w_{\text{thermal}}/2$. Thus, $\tau = 2H/w_{\text{thermal}}$. The terminal speed is then given in terms of itself:

$$w_{\text{thermal}} = \frac{(\rho_{\text{thermal}} - \rho_{\text{ambient}})g}{\rho_{\text{thermal}}} \frac{2H}{w_{\text{thermal}}} ,$$

and thus

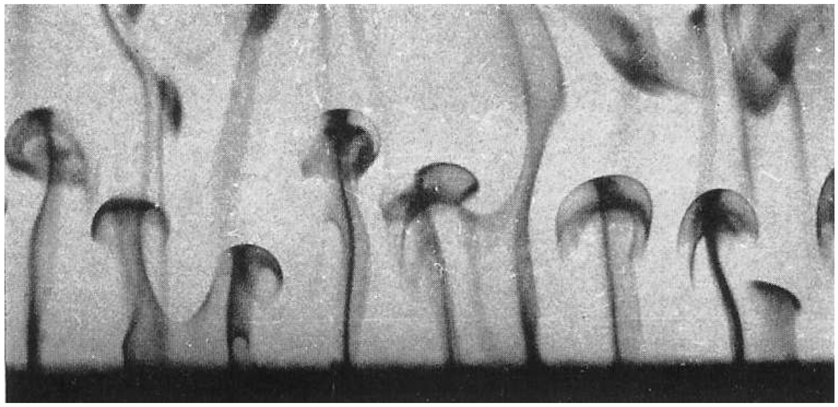

Figure 7.3: Thermals rising in water above a heated horizontal plate. (From Sparrow *et al.*, 1970)

$$w_{\text{thermal}}^2 \;=\; 2gH\,\frac{\rho_{\text{thermal}} - \rho_{\text{ambient}}}{\rho_{\text{thermal}}}. \tag{7.2}$$

Note that the preceding equation could also have been derived by stating conservation of energy: The kinetic energy acquired by the thermal is balanced by an equal reduction in potential energy.

Because the density anomaly is due to a temperature difference, the previous expression can also be expressed as

$$w_{\text{thermal}}^2 \;=\; 2\alpha g H\,(T_{\text{thermal}} - T_{\text{ambient}}), \tag{7.3}$$

after using the thermal's temperature as the reference temperature in the equation of state.

Vertical motion carrying a temperature anomaly is conveying a heat flux, so that a thermal of mass m_{thermal} carries a heat anomaly of $m_{\text{thermal}} C_p\,(T_{\text{thermal}} - T_{\text{ambient}})$, in which C_p is the fluid's heat capacity at constant pressure (1005 J/kg·°C for air and 4184 J/kg·°C for freshwater). If n is the number of thermals generated per unit horizontal area and per unit time, then the heat flux Q (heat conveyed per unit area and per unit time) is

$$Q \;=\; m_{\text{thermal}}\,C_p\,(T_{\text{thermal}} - T_{\text{ambient}})\,n.$$

The product $m_{\text{thermal}} n$ is the mass carried away by the thermals (per unit horizontal area and time) and is thus equal to $\rho_{\text{thermal}}\ f\ w_{\text{thermal}}$, in which f is the fraction of the horizontal plane occupied by thermals. Expressed in terms of the thermals' velocity, the heat flux is thus

$$Q \;=\; \rho_{\text{thermal}}\,C_p\,f\,w_{\text{thermal}}\,(T_{\text{thermal}} - T_{\text{ambient}}). \tag{7.4}$$

Using (7.3) to eliminate the temperature anomaly $T_{\rm thermal} - T_{\rm ambient}$, we obtain:

$$Q \;=\; \rho_{\rm thermal}\; C_p\; f\; \frac{w_{\rm thermal}^3}{2\alpha g H} \;,$$

which can be solved to obtain the thermals' velocity in terms of the heat flux:

$$w_{\rm thermal} \;=\; \left(\frac{2\alpha g H Q}{\rho_{\rm thermal}\; C_p\; f} \right)^{1/3} . \tag{7.5}$$

Dropping the dimensionless factors on the order of unity (2 and f) and replacing the thermals' density by the reference density ρ_0, we are led to define the following vertical velocity scale

$$w_* \;=\; \left(\frac{\alpha g H Q}{\rho_0 C_p} \right)^{1/3} , \tag{7.6}$$

which by virtue of its construction is representative of (*i.e.*, is a scale for) the vertical motion of the thermals effecting the convection.

A scale for the temperature fluctuations caused by the thermals can likewise be determined. Substituting in (7.4) expression (7.5) for the thermal's velocity, we can solve for the thermals' temperature anomaly:

$$\begin{aligned} T_{\rm thermal} - T_{\rm ambient} \;&=\; \frac{Q}{\rho_{\rm thermal}\; C_p\; f\; w_{\rm thermal}} \\ &=\; \left(\frac{Q^2}{2\alpha g H \rho_{\rm thermal}^2\; C_p^2\; f^2} \right)^{1/3} , \end{aligned} \tag{7.7}$$

which suggests the following scale for the temperature fluctuations:

$$T_* \;=\; \left(\frac{Q^2}{\alpha g H \rho_0^2\; C_p^2} \right)^{1/3} . \tag{7.8}$$

As thermals carry their heat anomaly across the height H of the system, they contribute to changing its averaged temperature T according to

$$\frac{dT}{dt} \;=\; \pm\; \frac{Q}{\rho_0\; C_p\; H} \;, \tag{7.9}$$

as the overall heat budget prescribes. The plus sign is chosen in the case of heating from below, and the minus sign in the case of cooling from above.

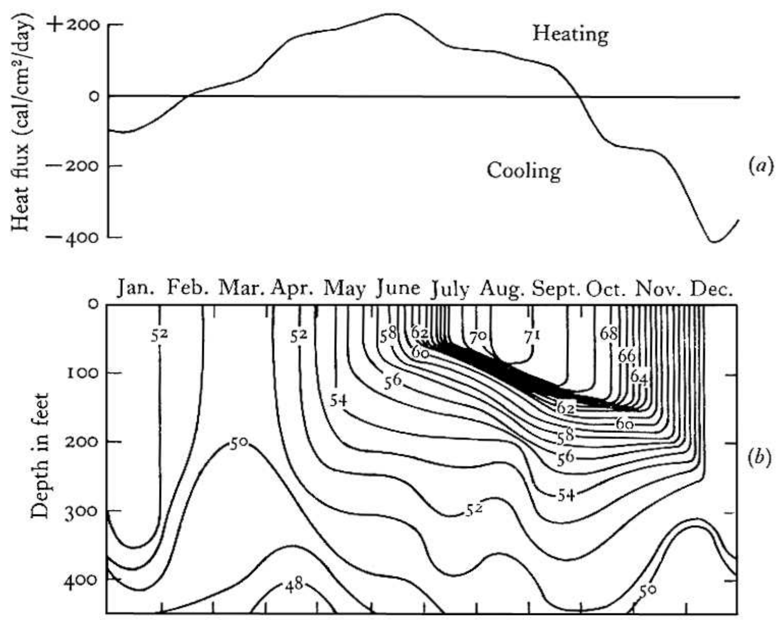

Figure 7.4: Cycle of seasonal heating and cooling in the subarctic Pacific Ocean: (a) Heat flux over the course of the year, and (b) the corresponding thermal structure of the upper ocean. Temperature contours are in degrees Fahrenheit. (From Tully and Giovando, 1963)

7.4 Penetrative Convection

The previous section considered a fully convecting system, in statistically steady state, except for the gradual decrease or increase of the overall temperature over time. In the environment, however, convection often operates against a pre-existing stratification, which it proceeds to erode. For example, a lake is typically stratified by summer's end and when comes autumn, cooling at the surface generates convection, which gradually destroys the underlying thermal stratification that had been built over the previous months. As cooling persists, the convective layer grows deeper and less of the stratification remains, until there is none left and convection engulfs the entire water column. In the ocean, the same seasonal process occurs, except that the convection is unable to reach the bottom by the time the heating season returns (Figure 7.4).

Likewise, the lower atmosphere cools during the night, and by morning the ground is covered by a layer of cold air. When the sun rises, the ground absorbs the solar radiation and heats the atmosphere from below, gradually erasing the night-time stratification. The convective layer, capped by the remnant of stratification is called the *atmospheric boundary layer* (ABL). The photograph in Figure 7.5 reveals

Figure 7.5: Thermals rising upwards and ending in clouds as indicators of atmospheric convection. (Photograph by Adrian Thomas)

the presence convection in the lower atmosphere under intense heating, manifested as wisps of condensation rising from the ground and ending in cloud puffs

Atmospheric penetrative convection has been simulated in the laboratory (Deardorff *et al.*, 1969). Figure 7.6 shows one such laboratory simulation of a thickening ABL. Note how nearly uniform is the temperature within the convective zone, which is not too surprising since vertical convective miotions create effective mixing. Curiously, the junction between the convective layer and the remaining, undisturbed stratification takes the form of a knee in which the temperature is locally lower than it was before convection reached that level. In other words, entrainment in the convective zone is first accompanied by a temperature decrease before the temperature increase expected under the action of the net heating of the system.

In each of these systems, the situation is time-dependent. The thickness of the convective layer increases over time, and the temperature within it decreases (lake, ocean) or increases (ABL). The process is called *penetrative convection.*

To study penetrative convection quantitatively, let us restrict the attention the case of a lake undergoing convection by surface cooling. At some intermediate time t, we define the thickness $h(t)$ of the convective region and $T(t)$, its well mixed temperature (Figure 7.7). The knee of the temperature profile at the base of the convective region is idealized as a discontinuity ΔT. Below the convective region ($z < -h$), the thermal stratification is still intact.

The initial temperature gradient and what remains of it below $z = -h$ is defined as

$$\bar{T}(z) \;=\; T_0 \;+\; \Gamma z, \tag{7.10}$$

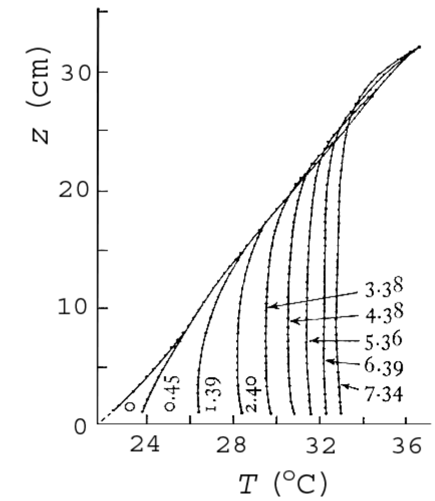

Figure 7.6: Successive vertical profiles of temperature in a laboratory experiment of penetrative convection, wherein a thermally stratified layer of water is heated from below. The labels mark the time in minutes. (From Deardorff *et al.*, 1969)

in which T_0 is the original surface temperature. The gradient $\Gamma > 0$ is taken as constant (linear stratification). To this gradient corresponds the stratification frequency N defined from

$$N^2 \ = \ \alpha g \, \frac{d\bar{T}}{dz} \ = \ \alpha g \Gamma \ > \ 0. \tag{7.11}$$

The temperature at the top of the remaining stratification is then given by

$$\bar{T}(-h) \ = \ T_0 \ - \ \Gamma h \tag{7.12}$$

and the temperature jump at the base of the convective zone is

$$\begin{aligned} \Delta T \ &= \ T \ - \ \bar{T}(-h) \\ &= \ T \ - \ T_0 \ + \ \Gamma h. \end{aligned} \tag{7.13}$$

The surface heat flux is denoted as Q (in W/m^2) and defined as positive upward (cooling). The overall heat budget (per unit horizontal area) from the time when the heat flux began ($t = 0$) to the present time ($t = t$) demands:

$$\int_{-h}^{0} \rho_0 C_p \, [\bar{T}(z) \ - \ T(t)] \, dz \ = \ \int_0^t \ Q(t') \, dt', \tag{7.14}$$

which states that the heat lost in the changing temperature profile between $z = -h$ and the surface is equal to the heat extracted through the surface during the intervening time. Performing the vertical integral, we obtain:

$$\rho_0 C_p \ \left[(T_0 - T) \ h \ - \ \Gamma \frac{h^2}{2} \right] \ = \ \int_0^t \ Q(t') \, dt'. \tag{7.15}$$

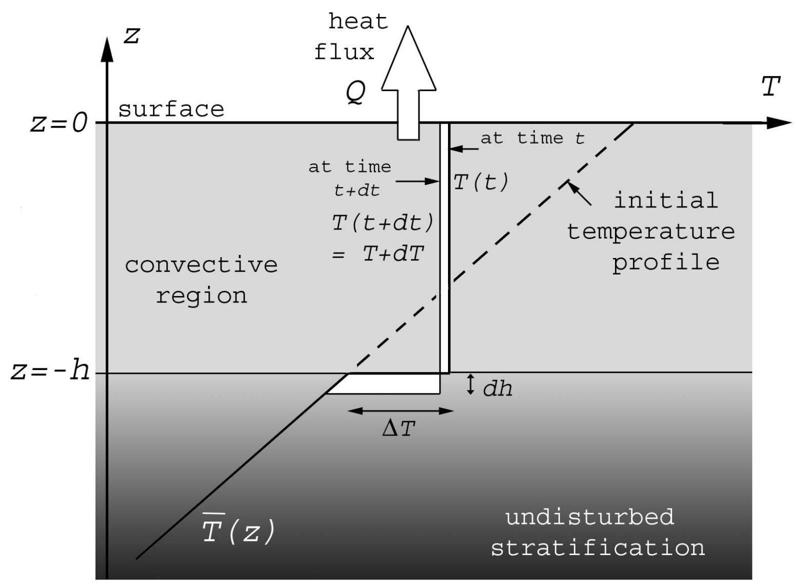

Figure 7.7: Sketch of penetrative convection, in the case of stratified body of water undergoing cooling from the surface.

The temporal derivative of this expression can be expressed as

$$\rho_0 C_p \left[\bar{T}(-h) \; \frac{dh}{dt} \; - \; \frac{d}{dt}(hT) \right] \; = \; Q, \tag{7.16}$$

which could also have been written by formulating the heat budget over a short time interval dt around time t (see Figure 7.7).

The preceding consideration gives one relation between the thickness h of the convective layer and its temperature T. A seond equation is necessary to obtain each separately. For this we turn to the mechanical energy budget.

The potential energy (per unit horizontal area) of the fluid in the vertical interval $-h \leq z \leq 0$ changes from

$$\begin{aligned} PE(t=0) \;&=\; \int_{-h}^{0} \rho(z,t=0)\; gz\; dz \;=\; \rho_0 g \int_{-h}^{0} 1 - \alpha\; [\bar{T}(z) - T_0]\; z\; dz \\ &=\; \rho_0 g \int_{-h}^{0} (z - \alpha\Gamma z^2)\; dz \;=\; \rho_0 g \left(-\frac{h^2}{2} - \alpha\Gamma \frac{h^3}{3} \right) \end{aligned}$$

to

$$\begin{aligned} PE(t) &= \int_{-h}^{0} \rho(z,t)\ gz\ dz\ =\ \rho_0 g \int_{-h}^{0} [1-\alpha\ (T-T_0)]\ z\ dz \\ &= \rho_0 g\ [1-\alpha\ (T-T_0)] \int_{-h}^{0} z\ dz\ =\ \rho_0 g\ \left(-\frac{h^2}{2}+\alpha\ (T-T_0)\ \frac{h^2}{2}\right), \end{aligned}$$

so that the change over time t since convection began is

$$\begin{aligned} \Delta PE &= PE(t)\ -\ PE(t=0) \\ &= \alpha\rho_0 g h^2\ \left(\frac{T-T_0}{2}+\frac{\Gamma h}{3}\right). \end{aligned} \tag{7.17}$$

We further make the drastic assumption — to be duly verified later — that the kinetic energy consumed by the sinking thermals is negligible. In other words, we assume that potential energy is barely consumed and merely rearranged. This is possible because the raising of the center of gravity by elevating heavier fluid and lowering lighter fluid can be balanced by the dropping of the center of gravity by overall thermal contraction of the fluid under net cooling. Mathematically, we simply write $\Delta PE = 0$, or

$$\frac{T_0-T}{2}\ =\ \frac{\Gamma h}{3}\ . \tag{7.18}$$

Solving for the instantaneous temperature T in the convective layer, we have

$$T\ =\ T_0\ -\ \frac{2\Gamma h}{3}\ , \tag{7.19}$$

which also provides the temperature jump at the base of the convective layer:

$$\Delta T\ =\ \frac{\Gamma h}{3}\ . \tag{7.20}$$

We now have two equations, (7.15) and (7.19), for the two unknowns, the instantaneous thickness of the convective layer, h, and its temperature, T. In terms of the cumulated heat flux $\int_0^t\ Q(t')dt'$, the solution is:

$$h(t)\ =\ \sqrt{\frac{6}{\rho_0 C_p \Gamma}\ \int Q(t')dt'} \tag{7.21}$$

$$T(t)\ =\ T_0\ -\ \sqrt{\frac{8\Gamma}{3\rho_0 C_p}\ \int Q(t')dt'}\ , \tag{7.22}$$

or, in terms of the stratification frequency N defined in (7.11),

$$h(t) = \sqrt{\frac{6\alpha g}{\rho_0 C_p N^2} \int Q(t')dt'} \tag{7.23}$$

$$T(t) = T_0 - \sqrt{\frac{8N^2}{3\alpha g \rho_0 C_p} \int Q(t')dt'} \; . \tag{7.24}$$

If the heat flux is constant, its integral over time is simply $\int Q(t')dt' = Qt$.

The task is not finished until we have verified that the kinetic energy pales in comparison to the terms in the potential energy. The kinetic energy (per unit horizontal area) is on the order of

$$KE \sim \frac{1}{2} \rho_0 \, h \, w_*^2,$$

where w_* is the thermal's vertical velocity scale given by (7.6) of the previous section. Thus,

$$\begin{aligned} KE &\sim \frac{1}{2} \rho_0 \, h \left(\frac{\alpha g h Q}{\rho_0 C_p} \right)^{2/3} , \\ &\sim \frac{1}{2} \left(\frac{\rho_0 \alpha^2 g^2 h^5 Q^2}{C_p^2} \right)^{1/3} . \end{aligned} \tag{7.25}$$

This is to be compared to one of the terms making the potential energy difference in (7.17), say

$$\Delta PE \sim \frac{\alpha \rho_0 g \Gamma h^3}{3} \; .$$

The ratio of kinetic to potential energy variations is then found to be

$$\begin{aligned} \frac{KE}{\Delta PE} &\sim \frac{3}{2} \left(\frac{Q^2}{\alpha \rho_0^2 g \Gamma^3 C_p^2 h^4} \right)^{1/3} \\ &\sim \frac{1}{(Nt)^{2/3}} \; . \end{aligned} \tag{7.26}$$

The time duration t of the convection (hours in the daily cycle to months in the seasonal cycle) is much longer than the stratification time scale $1/N$, which is typically on the order of seconds and minutes. So, indeed, kinetic energy is minuscule compared to the potential energy, and we were justified in neglecting its contribution to the mechanical energy budget. *A fortiori*, the decay rate of kinetic by frictional forces is, too, unimportant to the mechanical energy budget.

In a lake, it is not unusual for penetrative convection to reach the bottom by late winter, thereby resetting the bottom temperature and re-oxygenating the bottom waters. If cooling persists beyond this time, when no thermal stratification remains, convection proceeds from top to bottom according to the dynamics exposed in the previous section.

The previous analysis and equations were derived in the case of a water body being cooled from above. Everything applies to the case of the atmospheric boundary layer (ABL) being heated from below. One only needs to change the terminology from cooling to heating, downward to upward, etc.

7.5 Convection in a Rotating Fluid

Sinking and rising convective motions create pressure anomalies, which in the presence of the Coriolis force generate circling currents. These tend to confine laterally the convective motions into coherent vertical plumes, called *chimneys*. Examples in the ocean.

Horizontal scale and vertical velocity scale.

7.6 Convection Modeling

Text of section

Problems

7-1. Estimate the Rayleigh number characterizing the convection shown in Figure 7.4 during the month of September. (Use freshwater parameter values in the absence of seawater parameter values.)

7-2. You notice a buzzard soaring in a circling fashion and guess that it is taking advantage of the upward motion of a thermal. As you happen to have meteorological gear with you, including a radar profiler (to impress your friends) you determine that the buzzard is flying at an altitude of 60 m and that the temperature at the center of the bird's circle is 0.35°C higher than outside the thermal at the same height. What is the thermal's vertical velocity? Also, how old is this thermal?

7-3. Based on numbers shown in Figure 7.6, estimate the stratification frequency N (in 1/s) and heat flux Q (in W/m^2) used in the laboratory experiment performed by Deardorff and collaborators (1969).

7-4. The sun has been shining over the sea generating a sustained heat flux of 450 W/m^2 for three hours, which in turn is generating a growing convective layer. The non-convecting air above this layer is stratified with (potential) temperature increasing at the rate of 1.7°C every 100 m. Estimate the thickness of the convective layer after the three hours of sunshine and the increase in air temperature at sea level over the same time.

7-5. For the preceding problem, compare the velocity of rising thermals w_* to the growth rate dh/dt of the convective layer. Explain physically why it makes sense that the former is larger than the latter. [*Hint*: What would happen if the reverse were true?]

7-6. By the end of the summer, Wellington Reservoir (in Western Australia) is thermally stratified with temperature dropping with depth from 26°C at the surface to 14°C at 15 m depth. Below 15 m and all the way down to the bottom at 26 m, the water temperature is uniform at 14°C. Assuming a constant heat loss of 22 W/m^2 to the atmosphere during fall and early winter, how long does it take for penetrative convection to erase the summer stratification? Also, trace the vertical velocity w_* of the sinking thermals during the phase of penetrative convection and beyond (up to 5 months).

7-7.

7-8.