

## Chapter 2

# Physical Principles

SUMMARY: The object of this chapter is to establish the basic relationships that govern the physics of fluid motion, namely conservation laws for mass, momentum and energy. The control-volume approach is followed because it minimizes the use of mathematics and is well suited to a number of applications.

### 2.1 Control Volume

Perhaps the most direct and tangible way to state the physical principles that govern the motion of fluids is to formulate them in terms of budgets for finite portions of the fluid. For such enterprise, we first need to decide on two things: For what do we do a budget? And, where do we perform this budget? So, we select which physical quantity requires consideration (mass, momentum, energy or contaminant concentration) and define a certain volume as the basis for our analysis. The latter is called a *control volume*.

A control volume can be almost anything imaginable, a piece of atmosphere, a stretch of river, a whole lake, or even the entire troposphere. What matters most is that this volume be clearly defined (so we know unambiguously whether something is inside or outside it) and be practical (so the budget yields valuable information). There is no universal rule for choosing control volumes, but a common approach is to place the volume boundaries at locations where quantities are either already known or to be determined. In this fashion, the budget relates desired quantities to known quantities and helps determining the former in terms of the latter. Examples of practical control volumes are depicted in Figure 2.1.

Rarely will the control volume be bounded on all sides by impermeable boundaries. Thus, fluid enters the volume across some of its boundaries and leaves through some others, carrying with it mass, momentum, energy and any dissolved or suspended substance. When the fluid is regarded as a continuous medium, the meaningful quantity that describes such exchange between the control volume and its

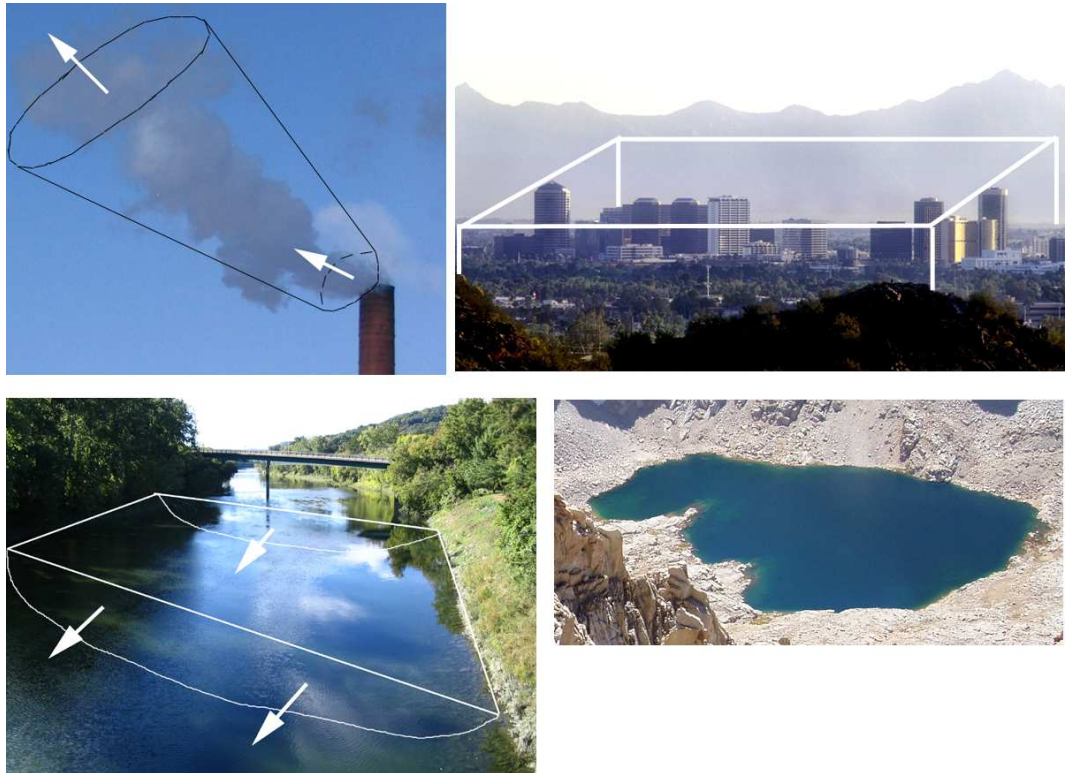


Figure 2.1: Examples of practical control volumes. [Photos clockwise from top right: smokestack in Hanover, New Hampshire, by the author; Phoenix, Arizona, ©Arizona-Leisure.com; lake in high Sierra Nevada by Dale S. Apgar; White River in Vermont by the author]

surroundings is the *flux*. The flux of any quantity (mass, momentum, energy, dissolved or suspended substance) is defined as the amount of that quantity that crosses a boundary per unit area and per unit time (Figure 2.2):

$$\text{flux} = q = \frac{\text{quantity crossing boundary}}{\text{boundary area} \times \text{time duration}}. \quad (2.1)$$

For example, if the quantity is mass, then the flux is a rate of mass per area and per time, to be expressed in units such as  $\text{kg}/\text{m}^2\text{s}$ .

This can also be written as:

$$q = \frac{\text{quantity}}{\text{volume of fluid}} \times \frac{\text{volume of fluid crossing boundary}}{\text{boundary area} \times \text{time duration}}. \quad (2.2)$$

The ‘quantity per volume of fluid’ can be denoted as  $c$ , the *concentration* of that quantity. If it is mass,  $c$  is mass per volume, that is, density and is denoted  $\rho$ ; if

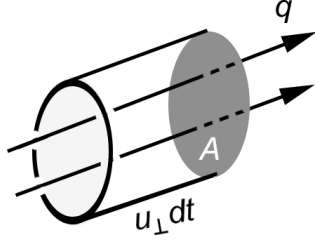


Figure 2.2: Definition of flux as flow of a substance through a boundary surface.

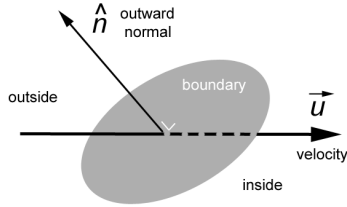


Figure 2.3: Oblique flux through a boundary. Only the normal component  $-\vec{u} \cdot \hat{n}$  of the velocity  $\vec{u}$  contributes to the flux.

it is momentum,  $c$  is mass times velocity (a vector) per volume, equal to density times velocity or  $\rho\vec{u}$ ; etc.

If the portion of the boundary under consideration has an area  $A$  and if  $u_{\perp}$  is the component of the flow velocity that is perpendicular to this area, then the amount of the quantity that crosses area  $A$  in the time interval  $dt$  is all contained in the fluid volume defined with  $A$  as its base and  $u_{\perp}dt$  as its length (Figure 2.2). Indeed, the fluid particles at distance  $u_{\perp}dt$  just arrive at the boundary when the time interval  $dt$  elapses, and all fluid particles ahead of them have crossed the boundary in the intervening time. The volume  $dV$  of fluid crossing the boundary is then  $dV = Au_{\perp}dt$  (= area times length), the amount of the quantity carried by that volume is  $cdV$  (amount per volume times volume), and the flux (amount carried per area per time) is:

$$q = \frac{cdV}{Adt} = \frac{cAu_{\perp}dt}{Adt} = cu_{\perp}. \quad (2.3)$$

Thus, the flux of any quantity through a boundary is equal to the product of the concentration of that quantity (amount per volume) by the velocity component perpendicular to the boundary.

If the fluid velocity is not exactly perpendicular to the boundary, only the normal component contributes to the flux because the parallel component causes no exchange across the boundary. To relate the direction of the velocity to the direction normal to the boundary, we define the unit vector that is perpendicular (normal)

to the boundary and directed outward, which we denote by  $\hat{n}$  (Figure 2.3). The component of the three-dimensional vector  $\vec{u}$  that is perpendicular to the boundary is then  $\vec{u} \cdot \hat{n}$ . However, as just defined, this component is positive if oriented along  $\hat{n}$ , that is, outward; to count the component as positive if it is directed inward, we need to change the sign:  $u_{\perp} = -\vec{u} \cdot \hat{n}$ . The definition of the flux is then generalized to

$$q = -c\vec{u} \cdot \hat{n}, \quad (2.4)$$

and the amount crossing an area  $A$  of the boundary per unit time is  $-c\vec{u} \cdot \hat{n}A$ .

We are now in a position to write a budget for a control volume. To account for the entire amount of the quantity under consideration, we write:

$$\begin{aligned} \text{Accumulation in control volume} &= \Sigma \text{ Imports through boundaries} \\ &- \Sigma \text{ Exports through boundaries} \\ &+ \Sigma \text{ Sources inside control volume} \\ &- \Sigma \text{ Sinks inside control volume.} \end{aligned}$$

Most often, such budget is written as a rate (that is, per unit time). The accumulation is then the difference between the amounts present in the control volume at times  $t$  and  $t + dt$ , divided by the time lapse  $dt$ . If we treat the control volume as a ‘bulk’ object, we do not distinguish separate portions and assign a single value or ‘mixed’ concentration  $c$  to the entire volume  $V$  of the fluid within the control volume. The amount present in the control volume is then simply  $cV$ , by definition of the concentration  $c$ . Should this lumping approach be unsatisfactory, it would then be necessary to define a series of smaller control volumes and to perform a separate budget analysis for each one.

Over the time duration  $dt$ , the accumulation rate is:

$$\text{Accumulation in control volume} = \frac{1}{dt} [(cV)_{t+dt} - (cV)_t],$$

and in the limit of an infinitesimal lapse:

$$\begin{aligned} \text{Accumulation in control volume} &= \frac{d}{dt}(cV) \\ &= V \frac{dc}{dt}, \end{aligned} \quad (2.5)$$

since the volume  $V$  of the control volume is fixed over time<sup>1</sup>.

Because the exports through boundaries can be counted as negative imports, a single summation suffices for all imports and exports. Per unit time, this sum is:

<sup>1</sup>Moving and/or deformable control volumes are extremely rare in environmental applications as budgets are almost always stated for regions rigidly constrained by land boundaries.

$$\begin{aligned}
\Sigma \text{ Imports minus exports through boundaries} &= \Sigma q_i A_i \\
&= \Sigma c_i u_{\perp i} A_i \\
&= -\Sigma c_i (\vec{u}_i \cdot \hat{n}_i) A_i, \quad (2.6)
\end{aligned}$$

where the summation over the index  $i$  covers all inlets and outlets.

The determination of the sources and sinks inside the control volume requires the specification of the quantity for which the budget is performed (mass, momentum, energy, etc.) and a knowledge of the mechanisms by which this quantity may be generated or depleted. For the moment, let us only subsume all sources and sinks in the single term  $S$ , equal to the net amount of the quantity that is generated inside the control volume per unit time:

$$\Sigma \text{ Sources minus sinks inside control volume} = S. \quad (2.7)$$

Now with all pieces together, the budget becomes:

$$V \frac{dc}{dt} = \Sigma c_i u_{\perp i} A_i + S, \quad (2.8)$$

where again the summation over the index  $i$  covers all inlets and outlets of the control volume.

In cases when system properties vary continuously inside the control volume and along its boundaries, the total amount of the quantity is to be expressed as a volume integral of its concentration, and the summation over all inlets and outlets must be replaced by an integration over all open portions of the surface enclosing the control volume, and we write:

$$\frac{d}{dt} \iiint c \, dV = - \iint c \vec{u} \cdot \hat{n} \, dA + \iiint s \, dV, \quad (2.9)$$

where  $dV$  is an elementary piece of volume,  $\hat{n}$  is the unit vector perpendicular to the piece of area  $dA$  of the boundary and is directed outward, and  $s$  is the source amount per volume. The triple integral covers the entire control volume, while the double integral covers its enclosing surface.

In the particular case of steady state, when the fluid flows through the system without any change over time, the time derivative vanishes and the budget reduces to:

$$\Sigma c_i u_{\perp i} A_i + S = 0, \quad (2.10)$$

or

$$\iint c \vec{u} \cdot \hat{n} \, dA = \iiint s \, dV. \quad (2.11)$$

These last equations are rather intuitive and could have been formulated directly, for they simply state that the net export of the quantity through boundaries is equal to the net source within the control volume.

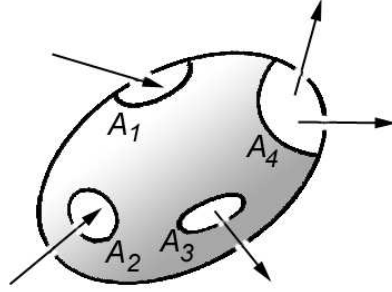


Figure 2.4: Mass budget for a control volume.

## 2.2 Conservation of Mass

A necessary statement is that mass be conserved. Indeed, “everything has to go somewhere”, and there is no source or sink for mass. In fluid systems, this means that the difference between the amount of mass that enters any control volume and the amount of mass that leaves it creates an equal accumulation or depletion of fluid inside that volume (Figure 2.4).

When the quantity under consideration is mass, the concentration  $c$  becomes mass per volume, or density, which we denote by  $\rho$  (units:  $\text{kg}/\text{m}^3$ ). Since there is no source or sink for mass ( $S = 0$ ), the budgets (2.8) and (2.9) become respectively:

$$V \frac{d\rho}{dt} = \Sigma \rho_i u_{\perp i} A_i , \quad (2.12)$$

and

$$\frac{d}{dt} \iiint \rho dV = - \iint \rho \vec{u} \cdot \hat{n} dA . \quad (2.13)$$

Keeping in mind that the density  $\rho$  does not vary much for natural fluids, we can write (see Section 1.3)  $\rho = \rho_0 + \rho'$ , where the first term is constant and the second variable but small compared to the first. This allows us to approximate the density of the fluid to the constant  $\rho_0$ , and the mass budget reduces to:

$$\Sigma u_{\perp i} A_i = 0, \quad (2.14)$$

or

$$\iint \vec{u} \cdot \hat{n} dA = 0, \quad (2.15)$$

depending on whether the boundary enclosing the control volume can be considered as a series of discrete inlets and outlets or needs to be treated with continuous variation. These last equations are none other than statements of conservation of volume. Indeed, when density (= mass per volume) is constant, volume becomes a

proxy for mass. This form of the mass-conservation principle is called the continuity equation.

A couple of remarks are in order. First, as intimated in Chapter 1, stratification is an essential ingredient of environmental fluid mechanics, and this implies that density variations, although small, can have significant dynamical effects. The astute student can then legitimately ask whether the neglect here of the variable part ( $\rho'$ ) of the density contradicts the expectation that stratification is important. The answer is that there is no contradiction because the importance of stratification is felt through the buoyancy forces, not through the mass budget. In other words, minor density differences are negligible for all practical purposes, except in combination with gravity. This last statement is generally known as the Boussinesq approximation, which will be discussed to a greater extent in the next chapter. Second, it is worth noting that the vanishing of the time derivative  $d\rho/dt$  when  $\rho$  is assimilated to the constant  $\rho_0$  does not imply in the least that we have restricted our attention to steady flows. Time derivatives of other variables, including velocity, may remain non-zero, so that the preceding continuity equation remains applicable to unsteady flows.

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**Example 2.1**

Consider Lake Nasser behind the Aswan High Dam in Egypt, which is fed by the Nile River (Bras, 1990). Annual averages of the upstream and downstream river flow rates are  $87 \text{ km}^3/\text{yr}$  and  $74 \text{ km}^3/\text{yr}$ , respectively, indicating a loss of water from the lake. Assuming that there is no ground seepage, the loss can only be caused by evaporation at the surface. Knowing that the lake surface area is  $1500 \text{ km}^2$ , we can determine the rate of evaporation by performing a water budget for the lake.

Density variations are quite negligible in this case, and a volume budget can take the place of a mass budget. Also, assuming that the data given above are for an average year, we can disregard any accumulation or depletion of water during the year, so that the year ends as it began; in other words, a steady state may be assumed. Under those conditions, we can apply budget (2.14) to the entire lake and obtain:

$$\text{River inflow} - \text{River outflow} - u_e A = 0,$$

where  $A$  is the lake's surface area, and  $u_e$  the vertical velocity at which water at the surface is lost to evaporation (also equal to the rate at which the water level would fall if the lake were not constantly replenished by the Nile River). It is the evaporation rate that we seek. Substituting the known values, we can solve for  $u_e$ :

$$\begin{aligned} u_e &= \frac{\text{River inflow} - \text{River outflow}}{A} \\ &= \frac{87 \text{ km}^3/\text{yr} - 74 \text{ km}^3/\text{yr}}{1500 \text{ km}^2} \\ &= 8.7 \text{ m/yr.} \end{aligned}$$

Thus, a surface layer of almost nine meters in thickness is lost to evaporation every year in Lake Nasser.

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### 2.3 Conservation of Momentum

Because fluid motions occur under the action of pressure and other forces, a necessary budget is that for momentum. The momentum of a piece of matter in movement is equal to the product of its mass by its velocity. In general, the velocity is a three-dimensional vector,  $\vec{u}$ , and on a per-volume basis, the concentration  $c$  of momentum becomes the vector  $\rho\vec{u}$ . Hence, the momentum budget forms a triple statement, one for each of the three Cartesian directions.

Newton's second law states that the unbalanced sum of forces causes an equal change in momentum. Therefore, forces act as sources and sinks of momentum. For a fluid inside a control volume, there are two kinds of forces, the superficial forces (such as pressure and viscous stress) acting on the fluid at the boundaries, and the body forces (such as gravity) acting throughout the volume.

The pressure  $p$  is defined as a normal force per unit area, positive inward. So, the pressure force on a surface of area  $A$  and with outward unit normal vector  $\hat{n}$  is  $\vec{F}_{\text{pressure}} = -p\hat{n}A$  (Figure 2.5). The gravitational force is mass times the gravitational acceleration, i.e.  $\vec{F}_{\text{gravity}} = \rho V\vec{g}$ , where  $\vec{g}$  is the earth's gravitational acceleration vector (equal to  $9.81 \text{ m/s}^2$  and directed vertically downward). Lumping all remaining forces as  $\vec{F}_{\text{other}}$ , the source term in the momentum budget is:

$$\vec{S} = -\sum p_j \hat{n}_j A_j + \rho V\vec{g} + \vec{F}_{\text{other}},$$

and the momentum budget follows from application of (2.8):

$$V \frac{d}{dt}(\rho\vec{u}) = -\sum \rho_i \vec{u}_i (\vec{u}_i \cdot \hat{n}_i) A_i - \sum p_j \hat{n}_j A_j + \rho V\vec{g} + \vec{F}_{\text{other}}. \quad (2.16)$$

In this expression, the summation over the index  $i$  covers all inlets and outlets, while the summation over the index  $j$  includes all surfaces of the control volume, because pressure is applied on the fluid even if there is no flow through the boundary.

If boundaries are to be treated with continuous variations, budget (2.9) is to be used, which becomes:

$$\frac{d}{dt} \iiint \rho\vec{u} dV = - \iint [\rho\vec{u}(\vec{u} \cdot \hat{n}) + p\hat{n}] dA + \iiint \rho\vec{g} dV + \vec{F}_{\text{other}}. \quad (2.17)$$

Because a uniform pressure exerts no net force, a constant value can be subtracted from all pressures. Doing this simplifies the analysis in a large number of problems. (When the subtracted value is the standard atmospheric pressure, the pressure is then called the *gage pressure*.)

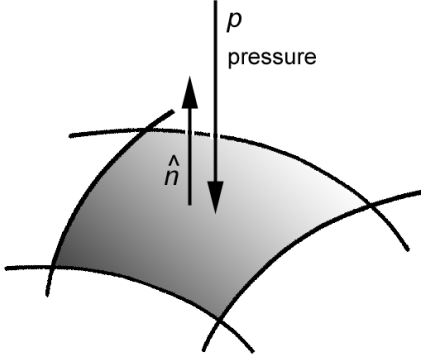


Figure 2.5: Pressure force on a surface and its relation to the normal unit vector.

As it was remarked in the previous sub-section, the density in environmental fluid flows varies only very moderately, and it is therefore permissible to replace the actual density  $\rho$  of the fluid by a constant value  $\rho_0$ , such as  $\rho_0=1.225 \text{ kg/m}^3$  for air at standard atmospheric conditions,  $\rho_0=999 \text{ kg/m}^3$  for freshwater, and  $\rho_0=1027 \text{ kg/m}^3$  for seawater. Density differences, however, must be retained in connection with the gravitational acceleration because of the buoyancy forces induced by gravity. Doing so, we may simplify the previous two forms of the momentum budget and write:

$$\rho_0 V \frac{d\vec{u}}{dt} = - \rho_0 \Sigma \vec{u}_i (\vec{u}_i \cdot \hat{n}_i) A_i - \Sigma p_j \hat{n}_j A_j + \rho V \vec{g} + \vec{F}_{\text{other}} . \quad (2.18)$$

or

$$\rho_0 \frac{d}{dt} \iiint \vec{u} dV = - \iint [\rho_0 \vec{u} (\vec{u} \cdot \hat{n}) + p \hat{n}] dA + \iiint \rho \vec{g} dV + \vec{F}_{\text{other}} . \quad (2.19)$$

We now present a series of examples to demonstrate how the momentum budget may be applied in specific circumstances and to show how diverse these applications can be.

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### Example 2.2

First, let us consider a jet exiting from a circular pipe at high speed and intruding in the same fluid at rest. This may be either an air jet in a calm atmosphere or a water jet in a resting water body. A practical control volume is one that cuts across the jet at two arbitrary sections, encloses the intermediate portion of the jet and a large portion of the surrounding fluid, large enough to extend to where the ambient fluid is perfectly still (Figure 2.6). We further assume that, despite obvious turbulent fluctuations, the overall properties of the jet remain unchanged over time (so that we may assume a statistically steady state), and that the ambient fluid is unstratified. If the jet is horizontal, we may further neglect the influence of gravity.

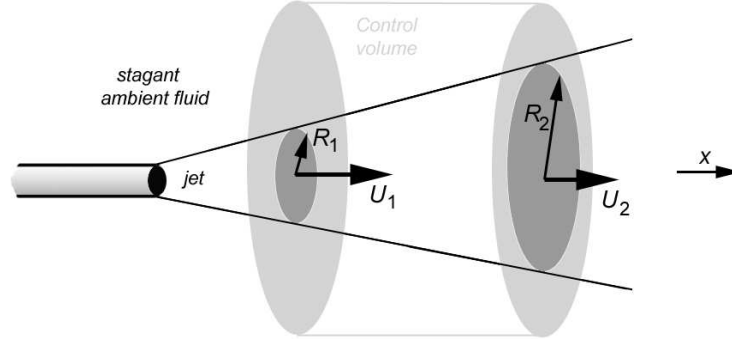


Figure 2.6: Momentum budget for a control volume surrounding a horizontal jet.

Because there is no transverse force and negligible transverse acceleration, the momentum budget in the transverse direction reduces to a self-cancellation of the pressure force, which implies that pressure must be uniform across the jet and thus equal to the ambient pressure. Further, because the ambient pressure must be uniform for lack of motion outside the jet, it follows that pressure is uniform not only in the transverse direction but also in the downstream direction, and there is no contribution of pressure to the downstream momentum budget. Application of (2.18) in the downstream direction then yields uniformity on momentum:

$$0 = + \rho_0 \iint_{\text{section 1}} u^2 dA - \rho_0 \iint_{\text{section 2}} u^2 dA,$$

which simplifies to:

$$\iint u^2 dA = \text{constant along the jet.}$$

This relationship implies that the speed at the center of the jet decreases downstream as the inverse of the jet radius:

$$U_2 = \frac{R_1 U_1}{R_2}. \quad (2.20)$$

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### Example 2.3

Next, let us determine how the pressure varies in a resting fluid under the action of gravity (Figure 2.7). For this, we consider a stagnant fluid of density  $\rho$  (possibly varying in the vertical direction) and, in it, a slice that is  $dz$  thick and has an arbitrary cross-sectional area  $A$ . The volume of that slice is  $V = Adz$ . There is no flow and no other force in the vertical besides gravity acting over the entire slice (of magnitude  $\rho gV$ ) and pressure acting on both top and bottom. The pressure at the top ( $p + dp$ ) is different from that at the bottom ( $p$ ) because the bottom pressure has to overcome both the top pressure and the force of gravity.

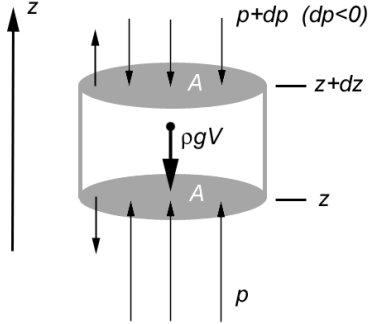


Figure 2.7: A layer of fluid in hydrostatic equilibrium.

The momentum budget (2.18) applied to the vertical direction reduces to a balance of forces:

$$0 = - (p + dp)(+1)A - p(-1)A + \rho V(-g)$$

which, after division by  $V$ , can be expressed as

$$\frac{(p + dp) - p}{dz} = - \rho g,$$

since  $V = Adz$ . In the limit of a vanishing thickness ( $dz \rightarrow 0$ ), the left-hand side becomes the  $z$ -derivative, and we have:

$$\frac{dp}{dz} = - \rho g. \quad (2.21)$$

This equation, which expresses how the pressure varies in the vertical across a resting fluid, is called the *hydrostatic balance*.

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#### Example 2.4

Finally, let us consider the flow of a river down its sloping bed (Figure 2.8). For simplicity, we assume that the channel cross-sectional area  $A$  and its wetted perimeter  $P$  are constant in the downstream direction.

The flow is not accelerating downward because the driving gravitational force is balanced by the retarding frictional force on the channel sides. Thus, for a control volume intercepting a chunk of the river between two sections  $dx$  apart from each other, the downstream velocity distribution is identical to that at the upstream section, and the pressure distribution, too, remains unchanged from one section to the other. Furthermore, there is no change in time, and a steady state may be assumed. Under those conditions, budget (2.18) reduces to:

$$0 = - \rho[U(-U)A + U(+U)A] - p(-1)A - p(+1)A + \rho(Adx)(+g \sin \theta) - \tau_f(Pdx),$$

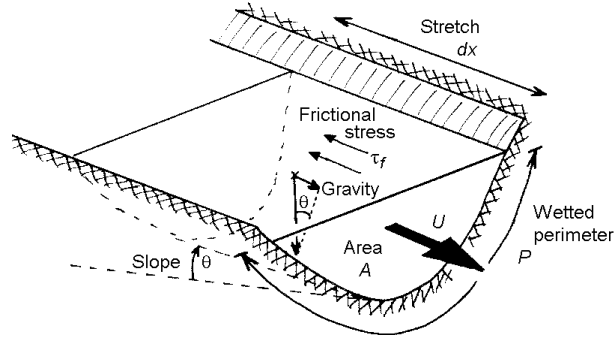


Figure 2.8: Momentum budget for a stretch of river along a sloping channel.

where  $U$  and  $p$  are the average velocity and pressure at both upstream and downstream sections,  $\rho$  is the water density,  $\tau_f$  the frictional stress acting along the surface  $Pdx$  of contact between water and channel bed, and  $\theta$  is the angle between the river bed and the horizontal (so that  $g \sin \theta$  is the projection of gravity in the downstream direction). The velocity and pressure terms cancel one another, leaving only the frictional force balancing the gravitational force. If we denote by  $S = \sin \theta$  the slope of the river bed and by  $R_h = A/P$  the so-called *hydraulic radius*, we obtain a relationship between stress and slope:

$$\tau_f = \rho g R_h S.$$

As we will see later, turbulence causes the frictional stress to be nearly proportional to the square of the velocity, i.e.

$$\tau_f = \rho C_D U^2,$$

where  $C_D$  is a dimensionless drag coefficient. Elimination of the stress between these last two relationships then yields the velocity as a function of the slope:

$$\begin{aligned} U &= \sqrt{\frac{g R_h S}{C_D}} \\ &= \text{constant} \sqrt{R_h S}. \end{aligned} \quad (2.22)$$

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There are occasions when the control volume reaches to a solid boundary beyond which the pressure is known. In such case, it is practical to extend the control volume to include this solid boundary, because the boundary pressure is no longer the unknown pressure exerted by the fluid on the inside of the boundary but the known outside pressure. This can simplify the analysis significantly.

When we include a solid boundary in the control volume, attention must be paid to any connection with other solid objects not included in the control volume,

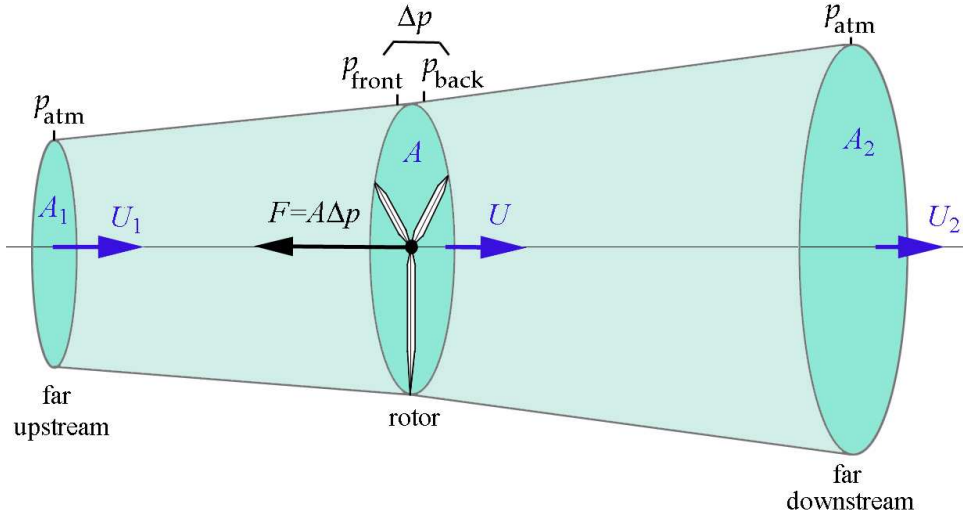


Figure 2.9: Wind subject to a braking force  $F$  as it blows through a windmill.

such as support arms and rivets. Forces in those connections must be explicitly incorporated in the momentum budget because they are outside forces acting on the control volume, as the following example illustrates.

### Example 2.5

Consider the wind upon encountering a windmill (Figure 2.9). Because it extracts energy from the wind, the rotor of the windmill exerts a braking force  $F$  on the wind, which is manifested by a difference  $\Delta p$  between a higher pressure  $p_{\text{front}}$  on the upstream face and a lower pressure  $p_{\text{back}}$  on the rear side (Figure 2.9). This force  $F = A\Delta p$ , where  $A$  is the surface area swept by the rotor, is related to the loss of momentum in the air flow.

For control volume, we define a tube of air laterally bounded by streamlines of the flow, defined by the circle swept by the rotor and extending both upstream and downstream to regions where the pressure can reasonably be assumed to be the unperturbed atmospheric pressure. We define three cross-sectional areas and three velocities:  $A_1$  and  $U_1$  upstream,  $A$  and  $U$  at the location of the rotor, and  $A_2$  and  $U_2$  downstream. Since no fluid can cross the lateral boundary of the tube, the mass entering the tube must be equal to that passing through the rotor and equal to that flowing downstream. Thus, mass conservation implies:

$$\rho_0 A_1 U_1 = \rho_0 A U = \rho_0 A_2 U_2,$$

from which we deduce the upstream and downstream cross-sectional areas in terms of the rotor area and the velocities:

$$A_1 = \frac{U}{U_1} A \quad A_2 = \frac{U}{U_2} A. \quad (2.23)$$

The momentum balance is quite simple because we can assume steady state (no time derivative), a uniform pressure along the periphery, and take a horizontal direction (no gravity term). If  $F$  is the force exerted by the windmill rotor on the airflow (naturally a retarding force from the perspective of the fluid), the momentum budget (2.18) can be stated as:

$$0 = -\rho_0 [U_1(-U_1)A_1 + U_2(+U_2)A_2] - F,$$

and yields the force  $F$ :

$$F = \rho_0 (A_1 U_1^2 - A_2 U_2^2). \quad (2.24)$$

With (??), it can be expressed in terms of the velocity difference between upstream and downstream:

$$F = \rho_0 A U (U_1 - U_2). \quad (2.25)$$

Since the force  $F$  should be positive as defined, it follows that

$$U_2 < U_1$$

indicating that the braking force exerted by the windmill is obviously at the expense of momentum in the fluid. (This analysis is continued in Example 2.7 below.)

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## 2.4 Bernoulli Equation

An interesting and very important corollary of the momentum budget is the so-called Bernoulli equation, which is obtained when budget (2.18) is applied to an infinitesimal cylinder aligned with a streamline in a steady-state flow. Consider a segment of length  $ds$  along a streamline and, centered on it, a cylinder of radius  $R$  (Figure 2.10). Then, assume that the fluid is inviscid and incompressible and that its flow is steady, so that there is no variation over time.

Conservation of mass (2.14) requires that the amount of fluid exiting downstream through cross-section  $\pi R^2$  at speed  $U + dU$  be equal to the amount entering upstream through the same cross-sectional area  $\pi R^2$  but at speed  $U$ , plus any amount entering along the lateral boundary  $2\pi R ds$  of the cylinder, because there may be no accumulation of fluid in steady state:

$$(\pi R^2)(U + dU) = (\pi R^2)U + (2\pi R ds)U_n, \quad (2.26)$$

where  $U_n$  the component of the velocity perpendicular to the cylinder's side. Solving for this velocity component, we have:

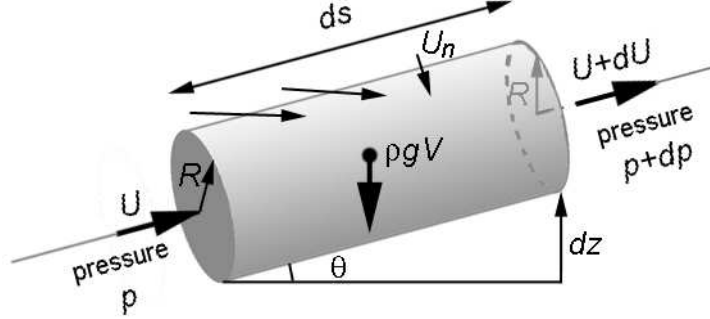


Figure 2.10: Mass and momentum budgets for a small tube aligned with the flow. The axis of the tube is a short stretch of streamline, and the cross-section is small and uniform.

$$U_n = \frac{R}{2} \frac{dU}{ds}. \quad (2.27)$$

Application of budget (2.18) for the momentum component aligned with the direction of the flow states that the exiting momentum is equal to the entering momentum through both upstream section and lateral boundary plus the sum of the pressure and gravitational forces:

$$\begin{aligned} \rho_0(\pi R^2)(U + dU)^2 &= \rho_0(\pi R^2)U^2 + \rho_0(2\pi R ds)UU_n \\ &+ (\pi R^2)p - (\pi R^2)(p + dp) \\ &- \rho(\pi R^2 ds)g \sin \theta, \end{aligned}$$

where we have taken the upstream pressure as  $p$  and the downstream pressure as  $p + dp$ . Also, in the gravitational term, we have been cautious to retain the full density  $\rho = \rho_0 + \rho'$  because of a possible buoyancy effect. The force of gravity is directed vertically and only its projection along the streamline needs to be considered, which is negative if the slope  $\sin \theta$  of the streamline is positive.

The side velocity  $U_n$  can be replaced by its expression (2.27), and we note that  $\sin \theta = dz/ds$  (see Figure 2.10), to reduce this momentum budget to:

$$\rho_0(\pi R^2)(U dU + dU^2) = -(\pi R^2)dp - \rho(\pi R^2)gdz.$$

Next, we drop the higher-order differential  $dU^2$  (much smaller than the first-order differentials in the limit toward zero), divide by  $\pi R^2 ds$  and gather all terms in the left-hand side, to obtain a more succinct expression:

$$\rho_0 U \frac{dU}{ds} + \frac{dp}{ds} + \rho g \frac{dz}{ds} = 0. \quad (2.28)$$

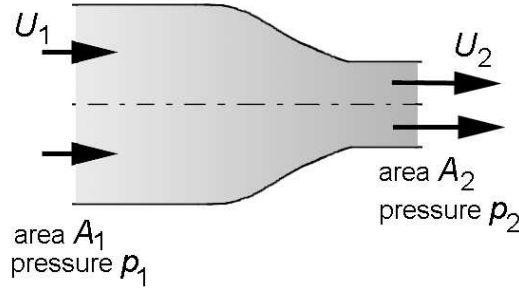


Figure 2.11: Application of the Bernoulli function.

If there is no stratification ( $\rho' = 0$  so that  $\rho$  is truly the constant  $\rho_0$ ) or, which is quite usual in a natural flow, if fluid parcels retain their individual density  $\rho$  while they flow (i.e.,  $\rho$  is constant along the chosen streamline although it may very well vary across streamlines), the preceding expression can be integrated from any point in the fluid to any other along the same streamline:

$$\rho_0 \frac{U_1^2}{2} + p_1 + \rho g z_1 = \rho_0 \frac{U_2^2}{2} + p_2 + \rho g z_2, \quad (2.29)$$

where subscripts 1 and 2 refer to any pair of upstream and downstream sections. Since the cross-section of the streamtube does not enter the above relation, we may easily take it to the limit of an infinitely narrow tube, that is, of a single streamline and state that the expression

$$B = \rho_0 \frac{U^2}{2} + p + \rho g z \quad (2.30)$$

is conserved *along a streamline* of a *steady, inviscid* flow, provided also that the fluid is unstratified or that its fluid parcels retain their individual density wherever they flow. The preceding expression is called the *Bernoulli function*.

---

### Example 2.6

As an application of the Bernoulli equation, consider the flow of sewage through a nozzle before exiting into a body of water (Figure 2.11). Inside the pipe leading to the nozzle, the cross-sectional area is  $A_1$ , the velocity  $U_1$ , and the fluid pressure  $p_1$ , while at the tip of the nozzle, the respective values are  $A_2$ ,  $U_2$  and  $p_2$  (ambient pressure).

Conservation of mass yields:

$$A_1 U_1 = A_2 U_2 \quad \rightarrow \quad U_1 = \frac{A_2}{A_1} U_2.$$

A basic momentum budget is not useful here because it would implicate the force exerted by the fluid onto the nozzle. So, we turn instead to the Bernoulli equation, which is applicable since we follow the flow from position 1 to position 2 and may neglect friction (assuming a short nozzle). Equation (2.29) provides

$$\rho_0 \frac{U_1^2}{2} + p_1 = \rho_0 \frac{U_2^2}{2} + p_2,$$

in which we neglected gravity by considering a horizontal nozzle. The pressure difference  $\Delta p = p_1 - p_2$  is thus given by

$$\Delta p = \frac{\rho_0}{2} (U_2^2 - U_1^2) = \frac{\rho_0 U_2^2}{2} \left( 1 - \frac{A_2^2}{A_1^2} \right) > 0. \quad (2.31)$$

This pressure difference is positive as long as  $A_2 < A_1$ . It tells what pipe pressure is required above the ambient pressure inside the receiving water body that is necessary to generate a desired exit velocity  $U_2$  (value required to achieve a certain level of mixing, a future topic).

With a connection now established between upstream and downstream, we can determine the force acting on the nozzle (in order to secure it properly). The momentum budget (2.18) yields:

$$0 = \rho_0 U_1^2 A_1 - \rho_0 U_2^2 A_2 + p_1 A_1 - p_2 A_2 - F.$$

A few algebraic steps then provide:

$$F = p_2 \Delta A + \frac{\rho_0 U_2^2}{2 A_1} \Delta A^2, \quad (2.32)$$

where  $\Delta A = A_1 - A_2$  is the difference in cross-sectional areas between pipe and nozzle tip.

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#### Example 2.7: The Betz limit on windmill performance

As a second application, we continue our consideration of the windmill (See start of the analysis in Example 2.5 and Figure 2.9) by applying the Bernoulli equation to both upstream and downstream sections of the control volume, up to but excluding the rotor.

Equation (2.30) gives for each section:

$$\begin{aligned} \text{Bernoulli in front:} \quad & \frac{1}{2} \rho_0 U_1^2 + p_{\text{atm}} = \frac{1}{2} \rho_0 U^2 + p_{\text{front}} \\ \text{Bernoulli in rear:} \quad & \frac{1}{2} \rho_0 U_2^2 + p_{\text{atm}} = \frac{1}{2} \rho_0 U^2 + p_{\text{back}}. \end{aligned}$$

The subtraction of these two equations provides a relation for the pressure difference  $\Delta p = p_{\text{front}} - p_{\text{back}}$

$$\begin{aligned} \frac{1}{2} \rho_0 (U_1^2 - U_2^2) &= p_{\text{front}} - p_{\text{back}} \\ \frac{1}{2} \rho_0 (U_1 + U_2)(U_1 - U_2) &= \Delta p. \end{aligned} \quad (2.33)$$

Since the force exerted by the windmill against the wind is  $F = A \Delta p$ , it follows that

$$F = \frac{1}{2} \rho_0 A (U_1 + U_2)(U_1 - U_2). \quad (2.34)$$

This same force  $F$  was already determined from the momentum budget, which included consideration of the passage of the air through the windmill rotor (what the Bernoulli principle excludes). Equation (2.25) then gave:

$$F = \rho_0 A U (U_1 - U_2), \quad (2.35)$$

and a new relation is found between the velocities:

$$\frac{1}{2} \rho_0 A (U_1 + U_2)(U_1 - U_2) = \rho_0 A U (U_1 - U_2)$$

from which follows that the air velocity  $U$  at the level of the rotor is the average between the upstream and downstream wind speeds:

$$U = \frac{U_1 + U_2}{2}. \quad (2.36)$$

From these elements, we are now in a position to determine how much power the windmill extracts from the wind. The power extracted is equal to the work done by the braking force per time and is thus the force  $F$  multiplied by the wind speed  $U$  at the rotor level:

$$\begin{aligned} \text{Power extracted} &= F U \\ &= \rho_0 A U^2 (U_1 - U_2) \\ &= \frac{1}{4} \rho_0 A (U_1 + U_2)^2 (U_1 - U_2). \end{aligned}$$

The reference is the amount of power available, which is the rate at which kinetic energy passes at the windmill site, if the windmill were not present. It is expressed as the upstream kinetic energy per volume  $(1/2)\rho_0 U_1^2$  times the volumetric flowrate  $AU_1$ :

$$\begin{aligned} \text{Power available} &= \frac{1}{2} \rho_0 U_1^2 \times AU_1 \\ &= \frac{1}{2} \rho_0 A U_1^3. \end{aligned}$$

The efficiency of the windmill, called the *coefficient of performance* and denoted  $C_p$ , is ratio of the ratio of the two:

$$\begin{aligned} C_p &= \frac{\text{Power extracted}}{\text{Power available}} \\ &= \frac{\frac{1}{4} \rho_0 A (U_1 + U_2)^2 (U_1 - U_2)}{\frac{1}{2} \rho_0 A U_1^3} \\ &= \frac{1}{2} (1 + \lambda)^2 (1 - \lambda), \end{aligned}$$

where the speed ratio  $\lambda = U_2/U_1$  measures the deceleration of the wind caused by the presence of the windmill.

The amount of deceleration produced by the windmill, and hence its performance, depends on its design and speed of rotation. We are thus unable to be more specific without plunging in the details of the airflow near the blades of the rotor, the number of blades, etc. However, as Albert Betz, a German engineer and colleague of Ludwig Prandtl, showed many years ago (Betz, 1920), there is a maximum achievable that is less than 100%. To show this, we simply consider the quantity  $\lambda$  as a positive variable less than one, and search for the maximum of the preceding expression. Straightforward algebra (taking the derivative and setting it to zero) shows that the maximum value that  $C_p$  may reach is  $16/27 = 0.593$ , for  $\lambda = 1/3$ .

It is concluded that no more than 59% of the wind energy can ever be extracted by a windmill. Large modern windmills with three slender blades attain coefficients of performance of 47%, which is quite remarkable.

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## 2.5 Equation of State

Thusfar, we have established two independent budgets, namely a budget for mass and a vectorial budget for momentum (equivalent to three separate budgets). These can be considered as four equations governing the three components of the velocity and pressure. Since fluid flow also implicates density, an additional equation is necessary. The equation of state provides the necessary relation.

In any fluid, the density is function of pressure and temperature:

$$\rho = \rho(p, T). \quad (2.37)$$

Water is nearly incompressible, and its density variation with pressure is negligible. Only the variation with temperature remains, which is known as *thermal expansion*. For air, which is compressible, the situation is somewhat delicate, but it suffices to note here that the compressibility effect can be eliminated by introducing a so-called *potential temperature* (details in Chapter 13) so that air density no longer depends on pressure but only on potential temperature. The modest range of temperature variations (actual temperature in water, potential temperature in the air) justifies a linear relationship:

$$\rho = \rho_0 [1 - \alpha(T - T_0)], \quad (2.38)$$

where  $\rho_0$  is the density at reference temperature  $T_0$  and  $\alpha$  is the coefficient of thermal expansion. Typical values for air are

$$T_0 = 15^\circ\text{C} = 288 \text{ K} \quad \rho_0 = 1.225 \text{ kg/m}^3 \quad \alpha = 3.47 \times 10^{-3} \text{ K}^{-1},$$

for freshwater

$$T_0 = 15^\circ\text{C} = 288\text{ K} \quad \rho_0 = 999\text{ kg/m}^3 \quad \alpha = 2.57 \times 10^{-4}\text{ K}^{-1},$$

and for seawater

$$T_0 = 10^\circ\text{C} = 283\text{ K} \quad \rho_0 = 1027\text{ kg/m}^3 \quad \alpha = 1.7 \times 10^{-4}\text{ K}^{-1}.$$

It remains, however, that the equation of state introduces a new variable, namely temperature, therefore demanding an additional equation. This is provided by the energy budget.

## 2.6 Conservation of Energy

In natural flows of water and air, the heat generated by the frictional dissipation of mechanical energy is negligible compared to the amounts of heat carried by the flow and redistributed by turbulence. In air flows, compressibility opens the way for an interchange between mechanical and internal (heat) forms of energy (because mechanical energy is transformed into internal energy during compression, and vice versa during decompression). But, the amount of energy converted under the natural variations of pressure and temperature in environmental air flows remains negligible. Thus, for all practical purposes, the budget for total energy can be conveniently reduced, for both water and air, to a heat-conservation budget.

Moreover, the narrow range of temperatures occurring in natural flows allows us to claim a linear relationship between the heat content  $Q$  of the fluid and its absolute temperature  $T$ , in the form

$$Q = mC_pT, \tag{2.39}$$

where  $C_p$  is the specific heat capacity (at constant pressure<sup>2</sup>). The corresponding *concentration*  $c$  according to the terminology of Section 2.1 is the heat content per unit volume, that is,  $\rho C_p T$ .

The heat budget then becomes an application of (2.8) to  $\rho C_p T$ .

$$V \frac{d}{dt}(\rho C_p T) = \Sigma (\rho C_p T)_i u_{\perp i} A_i + \text{heat sources}. \tag{2.40}$$

With the density being close to the reference and constant value  $\rho_0$  (as discussed in Section 1.3) and the heat capacity  $C_p$ , too, being nearly constant, the heat budget becomes the following equation for the temperature  $T$ :

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<sup>2</sup>For liquid water, it does not matter whether the heat capacity is that at constant volume or constant pressure, because both values are nearly the same, but for air, which is compressible, work is performed during compression and expansion, and the theory of thermodynamics tells us that pressure work is accounted for when using the enthalpy  $mC_p T$  instead of the internal energy  $mC_v T$ . Details are provided in Section 13.1.

$$V \frac{dT}{dt} = \Sigma T_i u_{\perp i} A_i + \frac{\text{heat sources}}{\rho_0 C_p}. \quad (2.41)$$

In the case of continuous variations across the control volume  $V$  and/or along the boundaries, as budget (2.8) is replaced by (2.9), so is (2.41) replaced by

$$\frac{d}{dt} \iiint T dV = - \iint T \vec{u} \cdot \hat{n} dA + \iiint \frac{q}{\rho_0 C_p} dV, \quad (2.42)$$

where  $q$  represents the heat source per unit volume inside the control volume.

For most environmental applications, the following values can be used for the specific heat capacity:

$$\text{Water : } C_p = 4184 \text{ J/kg K} \quad (2.43)$$

$$\text{Air : } C_p = 1005 \text{ J/kg K.} \quad (2.44)$$

Thus, on a per-mass basis, water can hold about four times as much heat as air for every added degree of temperature. Since water density is far larger than that of air (999 versus 1.225 kg/m<sup>3</sup>), the ratio is much larger on a volume basis: For the same increase in temperature, 1 m<sup>3</sup> of water takes 3,400 more heat than 1 m<sup>3</sup> of air.

Equations (2.41) and (2.42) are general. Often, the system under consideration is in steady state and includes no internal sources of heat. In such case, the equations reduce to:

$$\Sigma T_i u_{\perp i} A_i = 0 \quad (2.45)$$

$$\iint T \vec{u} \cdot \hat{n} dA = 0, \quad (2.46)$$

which merely state that the sum of all heat fluxes across the boundaries is zero so that no heat is gained or lost in the system.

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### Example 2.8

Electricity-producing power plants often use water from a nearby river for their cooling cycle. Suppose that a power plant needs to dispose of 300 MW of waste heat and plans to do so by using a portion of a nearby river with flow rate of 80 m<sup>3</sup>/s at ambient temperature of 12°C. After disposal of the waste heat, the temperature of the used water may not exceed 16°C. How much should the plant take from the river? And, once the used water is returned to the river, by how many degrees will the river water be heated?

To answer the first question, we apply budget (2.41) to the portion of the water used by the plant. Under steady-state conditions and with a single flow  $u_1 A_1 = Q_{\text{in}}$  of intake water at temperature  $T_{\text{in}}$  and an equal flow  $u_2 A_2 = Q_{\text{out}} = Q_{\text{in}}$  of water exiting at temperature  $T_{\text{out}}$ , we have

$$0 = Q_{\text{in}}T_{\text{in}} - Q_{\text{out}}T_{\text{out}} + \frac{300 \text{ MW}}{\rho_0 C_p},$$

which, with  $Q_{\text{out}} = Q_{\text{in}}$ ,  $T_{\text{out}} - T_{\text{in}} = 16^\circ\text{C} - 12^\circ\text{C} = 4 \text{ K}$ , provides

$$\begin{aligned} Q_{\text{in}} &= \frac{(300 \times 10^6 \text{ J/s})}{(999 \text{ kg/m}^3)(4184 \text{ J/kg K})(4 \text{ K})} \\ &= 17.94 \text{ m}^3/\text{s}. \end{aligned}$$

This leaves  $80 \text{ m}^3/\text{s} - 17.94 \text{ m}^3/\text{s} = 62.06 \text{ m}^3/\text{s}$  in the river, staying at  $12^\circ\text{C}$ .

Once discharged back into the river, the heated water creates a mixing situation with two inflows ( $17.94 \text{ m}^3/\text{s}$  at  $16^\circ\text{C}$  and  $62.06 \text{ m}^3/\text{s}$  at  $12^\circ\text{C}$ ) and one outflow ( $80 \text{ m}^3/\text{s}$  at the unknown temperature  $T_{\text{mix}}$ ). Application of budget (2.45) yields:

$$(80 \text{ m}^3/\text{s}) T_{\text{mix}} = (17.94 \text{ m}^3/\text{s})(16^\circ\text{C}) + (62.06 \text{ m}^3/\text{s})(12^\circ\text{C}),$$

$$T = 12.90^\circ\text{C}.$$

Thus, the disposal of waste heat by the power plant heats the river by nearly 1 degree.

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## Problems

- 2-1.** A dissolved mineral enters a  $2.3 \times 10^5 \text{ m}^3$  lake by a stream carrying a concentration averaging  $6.8 \text{ mg/L}$  ( $= 6.8 \text{ g/m}^3$ ). The annual flow of this stream into the lake is estimated at  $9.2 \times 10^4 \text{ m}^3$ , but evaporation over the  $4.4 \times 10^4 \text{ m}^2$  surface of the lake reduces the outflow stream past the lake to  $8.3 \times 10^4 \text{ m}^3$  per year. The outflow concentration of the dissolved mineral is  $7.0 \text{ mg/L}$ . What is the rate of evaporation from the surface (in centimeters of water per year)? And, is there a source or sink of the mineral inside the lake? What is the rate of this source or sink (in grams per day)?
- 2-2.** A  $200\text{-m}^3$  swimming pool has been overly chlorinated by accidentally dumping an entire container of chlorine, which amounted to 500 grams. This creates a chlorine concentration well in excess of the  $0.20 \text{ mg/L}$  that was the objective at the time. With which volumetric rate (in  $\text{L/min}$ ) of freshwater should the pool be flushed to decrease the concentration to the desired level in 36 hours? You may assume that the chlorine does not degrade chemically during those hours.

- 2-3.** A metropolitan area extending 20 km along a 10 km-wide valley is almost steadily ventilated by a wind blowing down the valley with a velocity of 18 km/h. There is no upwind pollution, and the thickness of the atmospheric boundary layer (in which vigorous mixing takes place) is 1000 m.

Inside this metropolis, road traffic and other polluting sources emit nitrogen oxides at a rate that varies with the hour of the day. A practical assumption is to take the emission equal to 4000 kg/h during morning commuter traffic lasting from 6 to 9 o'clock in the morning, then equal to 2000 kg/h until 9 o'clock in the evening, and 800 kg/h during the remaining hours of the day and night, namely from 9 o'clock in the evening to 6 o'clock the next day morning.

Neglecting chemical reactions in which nitrogen oxides may participate, trace the concentration of nitrogen oxides in the airshed hour by hour. Run these calculations over several days keeping in mind that the initial concentration of the next day is the ending concentration on the previous day. Eventually, a cyclical situation is achieved. Identify the minimum and maximum concentrations of that daily cycle.

- 2-4.** Shortly after a torrential rain, the water level in a reservoir behind a dam rises at the rate of 1 cm per hour while the water allowed to pass beyond the dam remains set at 1500 m<sup>3</sup>/hour. Knowing that the reservoir surface area is  $7.2 \times 10^4$  m<sup>2</sup>, determine the current rate of inflow into the reservoir (in m<sup>3</sup> per hour).

If this rate of inflow is expected to persist for three days but the reservoir level may not be allowed to rise by more than a total of 40 cm, more water needs to be released at the dam than at the current rate. What is the minimum rate of water that the dam operator should let pass for those three days?

- 2-5.** A  $8.3 \times 10^5$  m<sup>2</sup> reservoir needs to be partially drained to perform maintenance work along its shore. What should be the withdrawal rate at the dam to lower the water level by 50 cm in 24 hours?

Just as the process is getting underway, however, a heavy rain dumps 2.1 cm of precipitation per hour and is expected to last 12 hours. What should be the amended withdrawal rate maintained over 24 hours to achieve the desired drop of 50 cm in 24 hours?

- 2-6.** The Rhine River at the level of Karlsruhe, Germany, has a nearly rectangular channel that is 190 m wide. During an episode of relatively high flow, the water depth was measured at 4.2 m and the volumetric flow rate estimated at 1370 m<sup>3</sup>/s. If the downstream channel slope is  $6 \times 10^{-4}$ , estimate the values of the frictional stress (in N/m<sup>2</sup>) and of the drag coefficient.

- 2-7.** A stream flows down a smooth bed with slope of 0.5 m per km. The cross-section of the channel bed is nearly parabolic and uniform in the downstream direction. At some location, measurements give a surface width of 12 m, a center depth of 0.80 m and an average velocity of 0.35 m/s. Assuming no loss

to friction because the channel bed is smooth, predict the water depth and average velocity 10, 20, 50 and 100 m downstream from this point. Is the speed of the water increasing or decreasing as water flows downhill?

- 2-8.** For the nozzle application (Example 2.6), derive Equation (2.32) from the preceding equations.
- 2-9.** At the beginning of Section 2.6 it is stated that heat dominates over other forms of energy. Check the validity of this assertion by comparing kinetic energy ( $mU^2/2$ ) and potential energy ( $mgH$ ) to changes in internal energy ( $mC_p\Delta T$ ) for an arbitrary mass  $m$  of fluid in the prototypical cases:
- Air parcel with  $U = 10$  m/s,  $H = 1000$  m and  $\Delta T = 10^\circ\text{C}$
  - Water parcel with  $U = 2$  m/s,  $H = 10$  m and  $\Delta T = 5^\circ\text{C}$ .
- 2-10.** University Lake<sup>3</sup> near Carrboro in North Carolina (USA) has a surface area of  $8.62 \times 10^5$  m<sup>2</sup> and a volume of  $2.23 \times 10^6$  m<sup>3</sup>. One spring, during a series of clear and warm days, the lake received from the sun an average of 123 W/m<sup>2</sup>. Estimate the rise in surface level generated by thermal expansion of the water in the lake. In your opinion, is the process of thermal expansion important compared to other processes capable of changing the water level, such as precipitation, evaporation and variable river discharge?
- 2-11.** In the lowest 10 km of the atmosphere, the air temperature decreases vertically by about 1°C for every 100 meters of height. Using the hydrostatic balance (2.21) and the equation of state (2.38), determine the pressure difference between ground level and an altitude of 2000 m.

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<sup>3</sup>Numbers in his problem were kindly provided by Thomas Shay at the University of North Carolina whose assistance is graciously acknowledged.