

Algebraic Connectivity Ratio of Ramanujan Graphs

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Abstract—In this paper, we explore spectral properties of a class of regular Cayley graphs known as Ramanujan graphs and prove that the ratio of their algebraic connectivity to that of regular lattices grows exponentially as $O(n^\gamma)$ with $\gamma = 1.84 \pm 0.05$ for networks with average degree of $O(\log(n))$. Explicit construction algorithms exist for Ramanujan graphs that create regular graphs with especial degree and scale that depend on a pair of prime numbers. We introduce a randomized algorithm for construction of a class of fast regular graphs called quasi Ramanujan graphs. These graphs are obtained from finite number of degree balancing operations on Watts-Strogatz small-world networks that are irregular graphs. We show that quasi Ramanujan graphs share similar combinatorial optimality spectral properties as Ramanujan graphs and are not restricted to especial choices of degree and scale. A byproduct of this fact is that the algebraic connectivity ratio of quasi Ramanujan graphs grows exponentially in n as well. Numerical experiments are performed to verify our analytical predictions. Consensus algorithms converge extremely fast on networks with exponentially growing algebraic connectivity ratios.

Index Terms—random graphs, Ramanujan graphs, Cayley graphs, small-world networks, algebraic connectivity, ultrafast consensus

I. INTRODUCTION

*Algebraic connectivity*¹ of graphs has recently emerged as an important performance criterion for the speed of information processing and coordination of multi-agent complex networked systems and sensors networks. Some problems in which algebraic connectivity plays a central role include consensus problems [22], [29], [24], [11], [26], [4], [20], [7], synchronization of coupled oscillators [27], flocking and swarming [19], [9], belief propagation and distributed hypothesis testing [21], and distributed filtering and data fusion in sensor networks [16], [23], [6], [18], dynamic graphs [14], [30], small-world networks [17], and Cayley graphs and expanders [1], [2], [5].

In this paper, we explore the growth rate of algebraic connectivity of Ramanujan graphs and a novel class of fast regular graphs compared to regular lattices on a ring (or nearest-neighbor type graphs). *Ramanujan graphs* are special class of *regular Cayley graphs* that are introduced by Lubotzky, Phillips, and Sarnak [13] in number theory. Since then, they have appeared in a variety of applications including the design of expander codes [25] (also known as LDPC codes) in information theory and expander graphs in computer science. The purpose of this paper is to demonstrate that both Ramanujan graphs and quasi Ramanujan graphs (to

be defined) are extremely fast graphs for solving consensus problems in complex networks compared to nearest-neighbor proximity graphs.

We introduce a notion called *algebraic connectivity ratio* that measures the ratio of the second largest Laplacian eigenvalue of a graph $G_{n,k}$ with average degree k and scale n to algebraic connectivity of a regular lattice with the same number of links. We provide an explicit formula for this eigenvalue ratio and prove that for Ramanujan graphs with $O(n \log(n))$ links, algebraic connectivity ratio grows exponentially as $O(n^\gamma)$ with $\gamma = 1.84 \pm 0.05$ over a large range of network scales.

For discrete-time consensus problems, Xiao & Boyd [29] solve the problem of optimizing the weights of a network with a *fixed* topology to achieve a low μ_1 (second smallest adjacency eigenvalue). This allows design of fast mixing Markov chains for a fixed topology [3]. Our goal is to address the combinatorial optimization version of the problem by finding multi-graph topologies that maximize algebraic connectivity of a network with nd links and average degree $k = 2d$. For especial values of n, k this problem was solved nearly two decades ago in the seminal paper [13] and the optimal solution is a Ramanujan graph. One of our contributions is to propose a simple algorithm for generation of quasi Ramanujan graphs from Watts-Strogatz small-world graphs that constitute high-performance suboptimal solutions of this problem (See Section V).

Here is an outline of the paper. In Section II some background on spectral properties of complex networks are provided. In Section III, we give an explicit formula for algebraic connectivity of regular lattices. In Section IV, we prove the growth rate of algebraic connectivity of Ramanujan graphs. In Section V, a randomized algorithm is introduced for generation of quasi Ramanujan graphs. Finally, concluding remarks are given in Section VI.

II. SPECTRAL PROPERTIES OF COMPLEX NETWORKS

Let $A = [a_{ij}]$ be the adjacency matrix of a graph $G = (V, E)$ with non-negative integer elements². Let $d_i = \sum_j a_{ij}$ denote the *degree* of node i and set $D = \text{diag}(d_1, \dots, d_n)$. The Laplacian of G is defined as follows:

$$L = D - A \quad (1)$$

The average degree of G denoted $\bar{d} = \frac{1}{n} \sum_i d_i$ is a measure of density of graphs with 0-1 weights. The size of a graph $|E| = n\bar{d}/2$ is directly determined by its average degree and

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¹Algebraic connectivity of a graph is the second largest eigenvalue of its Laplacian matrix.

²Throughout this paper, all adjacency matrices satisfy this property unless stated otherwise.

scale n . A graph is called a *complex network* for large n 's. In this paper, we are interested in spectral properties of complex networks with symmetric weights $A = A^T$.

Let us denote the eigenvalues of A by

$$\mu_0 \geq \mu_1 \geq \dots \geq \mu_{n-1}$$

and the eigenvalues of L by

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$$

to comply with the standard notation in spectral graph theory.

A graph G is called *k-regular* if all of its nodes have degree k . Spectral properties of adjacency and Laplacian matrix of *k-regular* graphs are closely connected via the following identity:

$$\lambda_l = k - \mu_{l-1}. \quad (2)$$

In other words, the Laplacian spectral properties of regular graphs follow directly from the spectral features of their adjacency matrix. This is due to the fact that for a *k-regular* graph

$$L = kI_n - A$$

where I_n denotes the identity matrix.

The second largest Laplacian eigenvalue λ_2 , or *algebraic connectivity*, of a connected graph appears in a number of applications including the speed of convergence of an *average-consensus algorithm*:

$$\dot{x}_i = \sum_{j \in N_i} a_{ij}(x_j - x_i) \quad (3)$$

where $N_i = \{j : a_{ij} \neq 0\}$ denotes the set of *neighbors* of node i . The collective dynamics of n agents applying this consensus mechanism is

$$\dot{x}(t) = -Lx(t), \quad x(0) \in \mathbb{R}^n.$$

The state $x_i(t)$ of every agent asymptotically converges to $\bar{x}(0) = \frac{1}{n} \sum_i x_i(0)$. The convergence analysis of this system can be found in [20], [22] for both discrete- and continuous-time with a detailed discussion of algebraic connectivity of graphs and digraphs.

Algebraic connectivity of a network can be used as a measure of *network speed*. The main objective of this paper is to find network topologies that are extremely fast for solving consensus problems. To compare the speed of a graph G with nk links, we use a base *k-regular* lattice on a ring with the same number of links.

III. ALGEBRAIC CONNECTIVITY OF REGULAR LATTICES

An (n, k) *regular lattice*, or $C_{n,k}$, is a *k-regular* graph with n evenly spaced nodes on a ring in which each node is connected to its k nearest neighbors (See Fig. 1).

The adjacency matrix A of an (n, k) regular lattice is a circulant matrix with k ones on each row. A *circulant matrix* is a matrix in the form

$$A = \begin{bmatrix} c_0 & c_1 & \dots & c_{n-1} \\ c_{n-1} & c_0 & \dots & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \dots & c_0 \end{bmatrix} \quad (4)$$

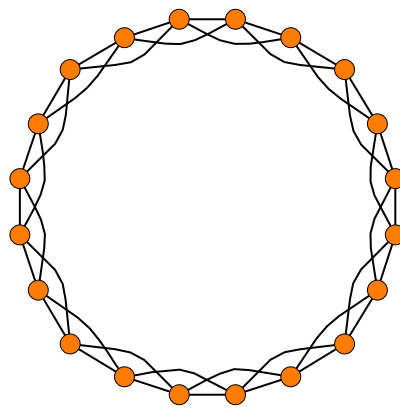


Fig. 1. A regular lattice on a ring with $(n, k) = (20, 2)$.

where each row is the cyclic right shift of the row above. Circulant matrices are special forms of Toeplitz matrices and their eigenvalues are the discrete Fourier transform (DFT) of their first row [12], [8].

Proposition 1. *Let A be the adjacency matrix of an (n, k) regular lattice with $k = 2d$. Then, the algebraic connectivity of this graph is given explicitly by*

$$\lambda_2 = k + 1 - \frac{\sin((k+1)\pi/n)}{\sin(\pi/n)} \quad (5)$$

Proof: For this *k-regular* graph, $\lambda_2 = k - \mu_1$. Defining $c_{-l} = c_{n-l}$ for $l > 0$, we have

$$\begin{aligned} c_{\pm l} &= 1, \forall l : 0 < l \leq d; \\ c_l &= 0, \forall l : d < l < n - d; \\ c_0 &= 0. \end{aligned}$$

Let μ_l be the l th eigenvalue of A , then

$$\mu_l = \sum_{k=0}^{n-1} c_k \exp\left(-\frac{j2\pi kl}{n}\right), \quad l = 0, 1, \dots, n-1 \quad (6)$$

with $j = \sqrt{-1}$. The last equation can be rewritten as

$$\mu_l = \sum_{k=-d}^d c_k \exp\left(-\frac{j2\pi kl}{n}\right); \quad (7)$$

$$= \sum_{k=1}^d c_k [\exp\left(-\frac{j2\pi kl}{n}\right) + \exp\left(\frac{j2\pi kl}{n}\right)]. \quad (8)$$

This gives the following set of eigenvalues for $C(n, k)$

$$\mu_l = 2 \sum_{k=1}^d \cos\left(\frac{2\pi kl}{n}\right). \quad (9)$$

From the last equation, $\mu_0 = 2d = k$ which corresponds to the trivial eigenvalue $\lambda_1 = 0$ of L . Furthermore, one obtains

$$\mu_1 = 2 \sum_{k=1}^d \cos\left(\frac{2\pi l}{n}\right)$$

after multiplying both sides of (9) by $\sin(\pi kl/n)$ and then use the identity $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta$, one gets

$$\mu_1 = \frac{\sin((2d+1)\pi/n)}{\sin(\pi/n)} - 1.$$

For a k regular graph, $\lambda_2 = k - \mu_1$, thus the algebraic connectivity of a k -regular lattice is

$$\lambda_2 = k + 1 - \frac{\sin((k+1)\pi/n)}{\sin(\pi/n)} \quad (10)$$

and the result follows. \square

Define $\theta_n = (k+1)\pi/n$. As $n \rightarrow \infty$, θ_n vanishes, thus for complex networks θ_n and π/n are both small. Clearly, for a fixed d or $d = O(\log(n))$, $\lim_{n \rightarrow \infty} \lambda_2 = 0$ which implies regular lattices get increasingly slower as n gets larger. This is a rather undesirable property of the speed of nearest-neighbor graphs.

Nearest-neighbors graphs arise in consensus-based information fusion in ad hoc sensor networks [26], [4], [16], [23], distributed hypothesis testing using belief propagation [21], and proximity graphs in flocking theory [19].

Remark 1. In information theory, the case of $d = O(\log(n))$ is of great significance for ad hoc wireless sensor networks due to limitations imposed on network capacity for network with more than $O(n \log(n))$ links or *average degree* of $O(\log(n))$ [10].

IV. RAMANUJAN GRAPHS AND ALGEBRAIC CONNECTIVITY RATIO

Ramanujan graphs³ are a special class of regular Cayley graphs with certain eigenvalue optimality properties that were introduced by Lubotzky, Phillips, and Sarnak (LPS) [13] (For a survey, see [15]). A k -regular graph $G_{n,k}$ of order n is called a *Ramanujan graph* if

$$\mu_1(G_{n,k}) \leq 2\sqrt{k-1}. \quad (11)$$

Let p and q be prime numbers and $p, q \equiv 1 \pmod{4}$. Then LPS offer an explicit $(p+1)$ -regular Cayley graph $X^{p,q}$ with $q(q^2-1)$ (or half of that) nodes that achieves the optimal upper bound in (12). On the other hand, due to pioneering work of Alon [1] on algebraic connectivity of graphs, we know that

$$\lim_{n \rightarrow \infty} \mu_1(G_{n,k}) \geq 2\sqrt{k-1}.$$

Based on Alon's result, algebraic connectivity of $G_{n,k}$ for large n 's satisfies the following inequality

$$\lambda_2(G_{n,k}) \leq k - 2\sqrt{k-1}. \quad (12)$$

Ramanujan graphs, including the explicit construction $X^{p,q}$, achieve the maximum value of λ_2 among all regular graphs.

Our objective is to explore the speed of $(2d)$ -regular Ramanujan graphs with $d = O(\log(n))$ (See Remark 1). For doing so, let us define the following notion.

³Also known as LPS graphs.

Definition 1. (algebraic connectivity ratio) Given a graph G with average degree $k \in \mathbb{Z}$, the *algebraic connectivity ratio* (ACR) (or *speed ratio*) is defined as

$$\rho(G) = \frac{\lambda_2(G)}{\lambda_2(C_{n,k})}. \quad (13)$$

Here is our main result on the growth rate of algebraic connectivity ratio of Ramanujan graphs.

Proposition 2. *Algebraic connectivity ratio of Ramanujan graphs $G_{n,k}$ with $k = 2d$ and $d = \lceil \log(n) \rceil$ (or $d = \lfloor \log(n) \rfloor \geq 1$) is explicitly given by*

$$\rho(G_{n,k}) = \frac{k - 2\sqrt{k-1}}{(k+1) - \frac{\sin((k+1)\pi/n)}{\sin(\pi/n)}}. \quad (14)$$

Furthermore, this ratio can be approximated up to an error $O(((k+1)\pi/n)^5)$ by

$$\hat{\rho}(G_{n,k}) = \frac{6(k - 2\sqrt{k-1})}{\pi^2(k+1)^3} n^2 \quad (15)$$

(the approximation error is of order 10^{-12} for $n = 10^4$).

Proof: The first part follows from Proposition 1. The second part is the main result that follows from expansion of $\sin(\theta_n) = \theta_n - \frac{1}{3}\theta_n^3 + O(\theta_n^5)$ after some straightforward calculations. \square

The main consequence of Proposition 2 is that the speed ratio of Ramanujan graph is $O(n^2)$ for a fixed k . Moreover, $\rho = O(n^\gamma)$ for $d = \log(n)$. Fig. 2 demonstrates the exponential growth rate of algebraic connectivity ratio of Ramanujan graphs for two cases of $k = O(\log(n))$ and $k = 6$ (note that $k = p+1$ for $p = 5$). As predicted in Proposition 2, the growth exponent of algebraic connectivity ratio for the case of a fixed k is $\gamma = 2$ and for the other case is $\gamma = 1.84 \pm 0.05$ that is close to 2. In both cases, the log-log plot is a straight line which proves the growth of ACR is exponential in n .

V. QUASI RAMANUJAN GRAPHS AND DEGREE BALANCING ALGORITHM

In this section, we introduce a randomize algorithm for explicit generation of *quasi Ramanujan graphs* that are regular weighted graphs with non-negative integer weights and average degree k that exhibit very similar spectral properties to Ramanujan graphs.

Let $S_{n,k} = G(p)$ denote a Watts-Strogatz small-world network obtained from *random rewiring* of links of $C_{n,k}$ with probability $p = 1$ [28]. $S_{n,k}$ is in general an *irregular graph* with the same average degree k as $C_{n,k}$. We present a randomized *degree balancing algorithm* that operates on random small-worlds $S_{n,k}$ and creates a k -regular graph. We refer to the output of the degree balancing (DB) procedure as *quasi Ramanujan graphs* and denote it by $Q_{n,k}$.

We show that the class of $Q_{n,k}$ regular graphs have the same extremal eigenvalue properties as Ramanujan graphs. The benefit of this algorithmic construction is that n and k no longer need to depend on prime numbers satisfying the

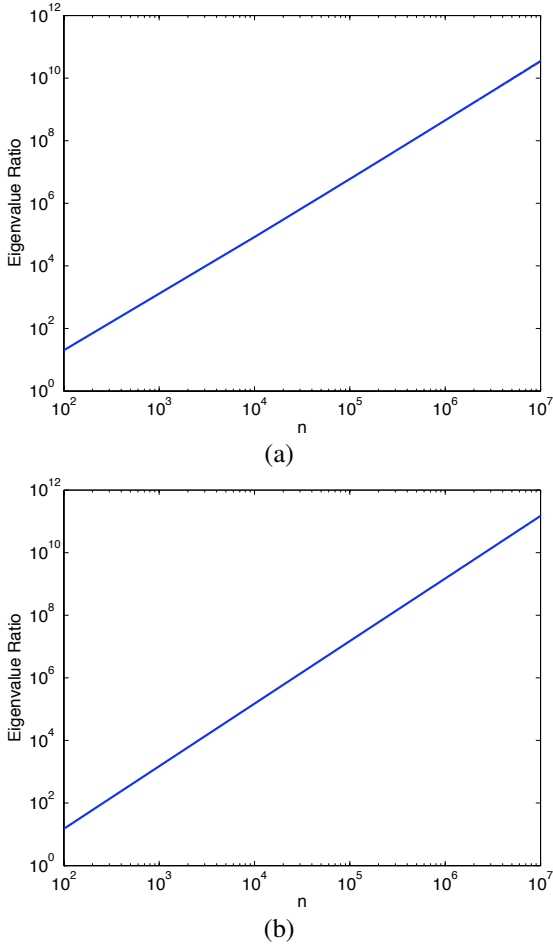


Fig. 2. Algebraic connectivity ratio of Ramanujan graphs: a) with $O(n \log(n))$ links and $\gamma = 1.84 \pm 0.05$ and b) with $O(nk/2)$ links and $k = 6$ and $\gamma = 2.0 \pm 0.0004$.

conditions in [13]. They can be chosen arbitrarily as long as $k = 2d$ is even.

Definition 2. (rich and poor nodes and degree gap) Let i_{max} and i_{min} be the indices of nodes with maximum and minimum degrees Δ_{max} and Δ_{min} . Then, i_{max} and i_{min} are called the *rich* and the *poor node* of the graph, respectively. The difference $\Delta_{gap} = \Delta_{max} - \Delta_{min} \geq 0$ will be called the *degree gap*.

A regular graph has a zero degree gap and an irregular graph has a positive degree gap.

Algorithm 1 takes some random neighbors of the rich nodes and gives them to the poor nodes until all nodes have the same degree. Symbolically, the degree balancing algorithm can be written as a *graphical dynamical system* [14]

$$X_{t+1} = F(X_t), X_0 = S_{n,k} \quad (16)$$

with a state X_t that is a *dynamic graph* and an initial state that is an irregular small-world network.

Proposition 3. *The degree balancing algorithm converges to a regular graph $Q_{n,k}$ in finite number of steps.*

Algorithm 1 Degree Balancing Algorithm

- 1: *Initialization:* At $t = 0$ set $X_t := S_{n,k}$ and $\Delta_{gap} = 1$.
 - 2: **while** $\Delta_{gap} > 0$ **do**
 - 3: *Find Rich and Poor Nodes:* Find nodes i_{max} and i_{min} of X_t .
 - 4: *Measure Degree Gap:* Compute Δ_{gap} .
 - 5: *Rewire:* Find a random neighbor j of the rich node and give this neighbor to the poor node by rewiring the link $\{j, i_{max}\}$ in X_t to $\{j, i_{min}\}$. Set X_{t+1} to the rewired graph.
 - 6: **end while**
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Proof: The degree gap function $\Delta_{gap}(t) = \Delta_{gap}(X_t)$ acts as a discrete-time Lyapunov function for system (16) as it either strictly decreases, or is zero. Since $\Delta_{max} \leq n - 1$ and $\Delta_{min} \geq 1$, initially the degree gap has an upper bound of $n - 2 < \infty$. The algorithm converges in at most Δ_{gap} steps and the final state of (16) must have a zero degree gap, i.e. $\exists N > 0 : X_t = Q_{n,k}, \forall t \geq N$. \square

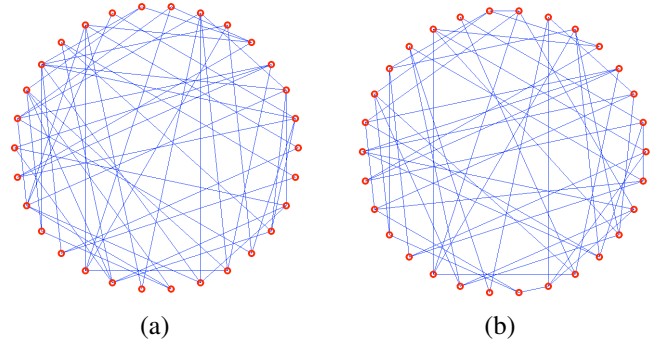


Fig. 3. Application of the degree balancing algorithm to a small-world graph in (a) that results in a quasi Ramanujan graph in (b). The two graphs are practically indistinguishable, but they have different spectral properties.

Fig. 3 shows an initial small-world graph $S_{30,4}$ and the quasi Ramanujan graph obtained from that.

Definition 3. (quasi Ramanujan graphs) Suppose Algorithm 1 converges in $N > 0$ steps. We formally define *quasi Ramanujan graphs* as the class of random k -regular graphs X_t at time $t = N$. In other words, a quasi-Ramanujan graph is the terminal state of graphical dynamical system (16) initialized with a quasi-random Watts-Strogatz small-world graph with a random rewiring probability of $p = 1$.

Apparently, the random end of the spectrum ($p = 1$) of Watts-Strogatz small-world graphs is explicitly used in construction of quasi Ramanujan graphs. In fact, initializing Algorithm 1 with something other than an quasi-random small world graph will simply not lead to emergence of graphs whose algebraic connectivity ratio is close to Ramanujan graphs. Here is our main conjecture:

Conjecture 1. *Any quasi Ramanujan graph $Q_{n,k}$ satisfies*

$$\mu_1(Q_{n,k}) \leq 2\sqrt{k-1}, \quad (17)$$

or $\lambda_2(Q_{n,k}) \geq k - 2\sqrt{k-1}$ for a sufficiently large n and $k = 2d$ with $d = O(\log(n))$.

In other words, if this conjecture holds, *quasi Ramanujan graphs have the same algebraic connectivity ratio as Ramanujan graphs*. The difference is that $Q_{n,k}$ is a weighted graph and not a graph with 0-1 weights. This difference has no negative effects in consensus problems but it makes a major difference in construction of *expander codes* [25] in information theory that result from Ramanujan graphs.

One can numerically examine this conjecture by measuring the performance r

$$r(Q_{n,k}) = \frac{\mu_1(Q_{n,k})}{2\sqrt{k-1}} \quad (18)$$

of quasi Ramanujan graphs and verifying how close it is to 1. Our experiments are summarized in Table I. As n increases, the performance r approaches the optimal upper bound of 1.

TABLE I
PERFORMANCE OF QUASI RAMANUJAN GRAPHS

n	$k = 2\lceil \log(n)/2 \rceil$	r (performance)
100	6	0.9393
200	6	0.9767
300	6	0.9698
400	6	0.9813
500	8	0.9817
750	8	0.9967
1000	8	0.9910
1250	8	0.9938

Let $G_{n,k}$ be a Ramanujan graph, then the difference between algebraic connectivity ratio of $G_{n,k}$ and $Q_{n,k}$ is

$$\frac{\Delta\rho}{\rho} = \frac{2(1-r)\sqrt{k-1}}{k-2\sqrt{k-1}} \approx 0.02$$

which vanishes as r approaches 1. This evidence suggests that quasi Ramanujan graphs are combinatorially *sub-optimal solutions* of the second eigenvalue maximization problem and their algebraic connectivity ratio is near %98 of the optimal solution. Considering the simplicity of generation of $S_{n,k}$ and $Q_{n,k}$, this is a satisfactory performance for a fast k -regular graph.

The byproduct of obtaining quasi Ramanujan graphs with a high performance is that defining the doubly stochastic matrix $P = I - \epsilon L(Q_{n,k})$ with $\epsilon = 1/k$ gives rise to *extremely fast mixing Markov chains*. Of course, P is the random walk matrix on a quasi Ramanujan graph.

It turns out that the second eigenvalue of $S_{n,k}$ changes slightly due to degree balancing. This implies that irregular small-worlds $S_{n,k}$ generated with rewiring probability of $p = 1$ must have an exponentially growing algebraic connectivity ratios. This is an issue that is open to further explorations in the future.

VI. CONCLUSIONS

We provided an explicit formula for algebraic connectivity of regular lattices $C_{n,k}$. In addition, we proved that

the algebraic connectivity ratio of Ramanujan graphs compared to $C_{n,k}$ grows exponentially in n with an exponent $\gamma = 1.84 \pm 0.05$ for networks with $O(n \log(n))$ links. We introduced a randomized degree balancing algorithm that transforms a Watts-Strogatz small-world network with $p = 1$ to a quasi Ramanujan graph $Q_{n,2d}$ with arbitrary parameters, high performance r , and algebraic connectivity ratio close to Ramanujan graphs as shown in Table I. The degree balancing algorithm takes some random neighbors from the rich nodes and gives them to poor nodes until all nodes have the same degree. Some connections to fast mixing Markov chains and random walks on Ramanujan graphs were discussed. We posed some open problems that need to be addressed in the future.

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