Fixed Point Controllers and Stabilization of the Cart-pole System and the Rotating Pendulum

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Abstract

In this paper, we consider stabilization of nonlinear systems in a special normal form as the cascade of a nonlinear subsystem and a linear subsystem. These systems do not possess any particular triangular structure. Despite this fact, we show how a backstepping type procedure applied to these systems naturally leads to a fixed point equation in the control input. We give conditions for well-posedness of these fixed point equations and show how these fixed points called Fixed Point Controllers (FPC) can be used for stabilization of cascade nonlinear systems. As special cases, we apply our results to semiglobal stabilization of two complex underactuated nonlinear systems, namely the cart-pole system and the rotating pendulum.

1 Introduction

In this paper, we consider stabilization of cascade nonlinear systems consisting of a nonlinear part and a linear part (i.e. chain of integrators). For nonlinear systems in a triangular strict feedback form, backstepping procedure can be applied. Here, we consider cascade nonlinear systems that do not possess any particular triangular structure. As a special form, we consider cascade of a nonlinear system with a double integrator. We show that a backstepping type procedure for nonlinear systems in this form naturally leads to fixed point equations in the control input. We provide sufficient conditions under which these fixed point equations are well-posed (i.e. have unique solutions). Then, we show how these fixed points can be used as stabilizing control laws for these cascade nonlinear systems. We call these fixed points, Fixed Point Controllers (FPC). The idea of using fixed point of an equation as a controller was first introduced by the authors in [4] for control of the beam-and-ball system. The main advantage of using fixed point controllers is to reduce the complexity of control design for a higher order system to control design for only a lower order nonlinear subsystem of the original system. Any method that somehow decouples the subsystems of the original system or partially linearizes it can be highly beneficial in further steps of control design using FPC's. We demonstrate this design strategy (i.e. reduction method) by applying our theoretical results to control design for two complex underactuated mechanical systems, namely the cart-pole system and the rotating pendulum (See Figures 1 and 3). Both of these systems are of high interest as testbeds for nonlinear control design among the researchers [8, 7, 11, 1, 9, 2]. In both of the examples we consider, the task is to stabilize the pendulum in its upright position at the origin from any initial condition. Under the condition that the pendulum cannot fall below the horizontal plane during the stabilization process. This sets a constraint on one of the state variables of the system. A global change of coordinates for decoupling of the dynamics of a single-input fourth-order nonlinear system (theorem 2.1) has been used to transform the dynamics of both the cart-pole system and the rotating pendulum to cascade of a second-order nonlinear system and a double integrator system. This decoupling method was first introduced in [3] by the authors and successfully used for control of the Acrobot. Semiglobal stabilization to a point equilibrium was achieved for both the cart-pole system and the rotating pendulum using fixed point controllers. In addition, for the cart-pole system global stabilization is achieved using Teel's nested saturations method [10]. The outline of the paper is as follows. First, we give some preliminary definitions and theorems. Then, we present our main results. Next, we give examples for our control design method with simulation results. Finally, we give concluding remarks.

2 Preliminaries

Notation. $|\cdot|$ denotes the Euclidean norm in R^n.

Notation. All compact sets are denoted by $\mathcal{K}$ with an appropriate subscript.

Definition 2.1. By a *sigmoidal function* $\sigma(s)$, we mean a smooth function that is bounded, strictly increasing with the property that $\sigma(0) = 0$ and $s\sigma(s) > 0$,

$\forall s \in R\backslash\{0\}$.

The following theorem is a key tool in decoupling of the dynamics of the cart-pole system and the rotating pendulum.

Theorem 2.1. *(Decoupling theorem) Consider the following system*

$$\begin{array}{rcl} \dot{q}_1 & = & p_1, \\ \dot{p}_1 & = & f_1(q,p) + g_1(q_2)u, \\ \dot{q}_1 & = & p_2, \\ \dot{p}_2 & = & f_2(q,p) + g_2(q_2)u, \end{array} \quad (1)$$

where $q = (q_1, q_2) \in R^2$, $p = (p_1, p_2) \in R^2$, f_i*'s and* g_i*'s are smooth functions, and* $g_2(q_2) \neq 0, \forall q_2 \in R$. *Then, the following global change of coordinates*

$$\begin{array}{rcl} z_1 & = & q_1 - \int_0^{q_2} g_1(s)/g_2(s)ds, \\ z_2 & = & p_1 - g_1(q_2)/g_2(q_2)p_2, \\ \xi_1 & = & q_2, \\ \xi_2 & = & p_2, \end{array} \quad (2)$$

decouples (q_1, p_1)*-subsystem and* (q_2, p_2)*-subsystem w.r.t.* u *and in new coordinates the dynamics of the system transforms into the normal form*

$$\begin{array}{rcl} \dot{z} & = & f(z, \xi_1, \xi_2), \\ \dot{\xi}_1 & = & \xi_2, \\ \dot{\xi}_2 & = & v, \end{array} \quad (3)$$

where $v = f_2(q,p) + g_2(q_2)u$ *and* $z = (z_1, z_2)^T$.

Proof. The proof is by direct calculation (See [3]). □

3 Main Results

In this paper, we consider systems in the following form

$$\begin{array}{rcl} \dot{z} & = & f(z, \xi_1, \xi_2), \\ \dot{\xi}_1 & = & \xi_2, \\ \dot{\xi}_2 & = & u, \end{array} \quad (4)$$

where $f(z, \xi_1, \xi_2) : R^m \times R^2 \to R^m$ is a smooth function with $f(0,0,0) = 0$. Suppose that there exists a smooth globally stabilizing state feedback law $\xi_1 = \alpha_1(z)$ for the z-subsystem when $\xi_2 = 0$, i.e. for the closed loop system

$$\dot{z} = f(z, \alpha_1(z), 0),$$

the origin $z = 0$ is globally asymptotically stable (GAS). Our main goal is to find classes of nonlinear systems in normal form (4) and certain conditions that allow us to find a semiglobally or globally stabilizing feedback law $u = k(z, \xi)$ for the composite system. To present our main result, first we need to make some assumptions.

Assumption 3.1. Suppose $\xi_1 = \alpha_1(z)$ with $\alpha_1(0) = 0$ is a smooth state feedback law that globally asymptotically (GAS) stabilizes the origin $z = 0$ for

$$\dot{z} = f(z, \xi_1, 0)$$

and let $V(z)$ be a smooth positive definite Lyapunov function for the closed loop system such that

$$\frac{\partial V(z)}{\partial z} f(z, \alpha_1(z), 0) < 0, \quad \forall z \neq 0$$

Assumption 3.2. Assume that $\xi_1 = \alpha_1(z)$ is an invariant manifold for the z-subsystem, i.e. the following fixed point equation

$$\xi_2 = \frac{\partial \alpha_1(z)}{\partial z} f(z, \alpha_1(z), \xi_2)$$

globally has a unique smooth solution

$$\xi_2^* = \beta_1(z)$$

that satisfies

$$\frac{\partial V(z)}{\partial z} f(z, \alpha_1(z), \beta_1(z)) < 0, \quad \forall z \neq 0$$

Assumption 3.3. Define

$$\begin{array}{rcl} \psi_2(z, \xi_1, \xi_2) & = & -\sigma(\xi_1 - \alpha_1(z)) + \dfrac{\partial \alpha_1(z)}{\partial z} f(z, \xi_1, \xi_2), \\ \psi_3(z, \xi_1, \xi_2, u) & = & -\sigma(\xi_2 - \psi_2) + \dfrac{\partial \psi_2}{\partial z} f(z, \xi_1, \xi_2) \\ & + & \dfrac{\partial \psi_2}{\partial \xi_1} \xi_2 + \dfrac{\partial \psi_2}{\partial \xi_2} u, \end{array}$$

and suppose the following fixed point equations

$$\begin{array}{rcl} \xi_2 & = & \psi_2(z, \xi_1, \xi_2) \\ u & = & \psi_3(z, \xi_1, \xi_2, u) \end{array} \quad (5)$$

globally have unique smooth solutions

$$\begin{array}{rcl} \xi_2^* & = & \alpha_2(z, \xi_1) \\ u^* & = & \alpha_3(z, \xi_1, \xi_2) \end{array} \quad (6)$$

Assumption 3.4. Assume the change of coordinates $\mu = T(z, \xi)$ given by

$$\begin{array}{rcl} \mu_1 & = & \xi_1 - \alpha_1(z) \\ \mu_2 & = & \xi_2 - \psi_2(z, \xi_1, \xi_2) \end{array}$$

is a global diffeomorphism with an inverse $\xi = T^{-1}(z, \mu)$ given by

$$\begin{array}{rcl} \xi_1 & = & \alpha_1(z) + \mu_1 \\ \xi_2 & = & \beta_2(z, \mu_1, \mu_2) \end{array}$$

where $\beta_2(z, 0, 0) = \beta_1(z)$.

The following two assumptions for boundedness of the solutions of cascade nonlinear systems are influenced by Sepulchre *et. al* [5].

Assumption 3.5. Suppose there exist $c_0, r_0 > 0$ such that the following condition hold

$$|\frac{\partial V(z)}{\partial z}| \cdot |z| \leq c_0 V(z), \quad \forall |z| > r_0$$

Assumption 3.6. Define

$$f_0(z,\mu) = f(z, \alpha_1(z) + \mu_1, \beta_2(z, \mu_1, \mu_2))$$

where $\mu = (\mu_1, \mu_2)^T$ and assume the interconnection term

$$\phi(z,\mu) = f_0(z,\mu) - f_0(z,0)$$

has global linear growth in z, i.e. there exist class-K functions $\omega_0(s), \omega_1(s)$ that are differentiable at the origin and

$$|\phi(z,\mu)| \leq \omega_0(|\mu|) + |z| \cdot \omega_1(|\mu|)$$

Here is our first main result:

Theorem 3.1. *(Fixed point backstepping I) Suppose Assumptions 3.1 through 3.6 hold. Then, $\xi_2 = \alpha_2(z, \xi_1)$ and $u = \alpha_3(z, \xi_1, \xi_2)$ globally asymptotically stabilize the origin for the (z, ξ_1)-subsystem of (4) and the composite system (4), respectively.*

Remark 3.1. Theorem 3.1 shows that a backstepping-type procedure for cascade nonlinear systems that are not in strict feedback form as in (4)) directly leads to fixed point equations in the control input of (z, ξ_1)-subsystem and (z, ξ)-system (i.e. the whole system). Conditions on existence and uniqueness of the solution of these fixed point equations determine the classes of nonlinear systems in non-feedback form that can be asymptotically stabilized. We show that for a rather large class of nonlinear systems at least semiglobal stabilization of (4) is possible.

Definition 3.1. We call a control law defined as the solution of a fixed point equation, a *fixed point controller*, provided that the fixed point equation is well-posed.

Proof. (theorem 3.1) Applying the change of coordinates

$$\begin{aligned} \mu_1 &= \xi_1 - \alpha_1(z) \\ \mu_2 &= \xi_2 - \psi_2(z, \xi_1, \xi_2) \end{aligned}$$

the dynamics of the μ-subsystem can be written as

$$\begin{aligned} \dot{\mu}_1 &= -\sigma(\mu_1) + \mu_2 \\ \dot{\mu}_2 &= -\sigma(\mu_2) \end{aligned}$$

It can be shown that $\mu = (\mu_1, \mu_2) = 0$ is GAS and locally exponentially stable (LES) for this system. Because $\mu = T(z, \xi)$ is a global diffeomorphism, one can solve $\mu_2 = \xi_2 - \psi_2(z, \xi_1, \xi_2)$ in ξ_2. Let $\xi_2 = \beta_2(z, \xi_1, \mu_2)$ be the solution of this equation. Note that at $(\mu_1, \mu_2) = 0$, $\beta_2(z, \xi_1, \mu_2) = \beta_1(z)$. The dynamics of the composite system can be written as

$$\begin{aligned} \dot{z} &= f(z, \alpha_1(z) + \mu_1, \beta_2(z, \mu_1, \mu_2)) \\ \dot{\mu} &= \eta(\mu) \end{aligned}$$

or

$$\begin{aligned} \dot{z} &= f_0(z, \mu) \\ \dot{\mu} &= \eta(\mu) \end{aligned}$$

where by assumption $f_0(z,\mu)$ has global linear growth in z. Also, given $\mu = 0$, $z = 0$ is GAS for

$$\dot{z} = f_0(z, 0) = f(z, \alpha_1(z), \beta_1(z))$$

from assumption 3.2. On the other hand, from assumptions 3.5 and 3.6 and the fact that $\mu = 0$ is GAS and LES for $\dot{\mu} = \eta(\mu)$, the solution of the z-subsystem is bounded for any solution of the μ-subsystem (see [5]). Therefore, based on Sontag's theorem for global stability of cascade nonlinear systems in [6], $(z, \mu) = (0, 0)$ is GAS for the closed loop system in (4). The proof of the GAS property of $(z, \xi_1) = 0$ for the (z, ξ_1)-subsystem follows as a special case of the preceding argument with $\mu_2(0) = 0$. □

The following is our second main result that is a semiglobal version of theorem 3.1 and has a crucial role in control design for nonlinear systems that we consider here.

Theorem 3.2. *(Fixed point backstepping II) Consider the following perturbed nonlinear system*

$$\begin{aligned} \dot{z} &= f_0(z, \xi_1) + g_0(z, \xi_1, \xi_2, \epsilon), \\ \dot{\xi}_1 &= \xi_2, \\ \dot{\xi}_2 &= u, \end{aligned} \tag{7}$$

where f_0, g_0 are smooth functions with $f_0(0) = 0$, $g_0(0) = 0$, $g_0(z, \xi_1, \xi_2, 0) = 0$, $g_0(0,0,0,\epsilon) = 0$, and all $\frac{\partial^2 g_0}{\partial z_i \partial \xi_j}$ vanish at the origin for $i = 1, \ldots, m$ and $j = 1, 2$. Suppose there exists a state feedback $\xi_1 = \alpha_1(z)$ with $\alpha_1(0) = 0$ such that for

$$\dot{z} = f_0(z, \alpha_1(z))$$

$z = 0$ is globally asymptotically and locally exponentially stable and let $V(z)$ be the Lyapunov function for this system with $\frac{\partial V(z)}{\partial z} f_0(z, \alpha_1(z)) < 0$, $\forall z \neq 0$. Decompose f_0 as the following

$$f_0(z, \xi_1 + \delta) = f_0(z, \xi_1) + h_0(z, \xi_1, \delta)\delta$$

and define

$$\begin{aligned} \psi_2(z, \xi_1, \xi_2, \epsilon) &= -\sigma(\xi_1 - \alpha(z)) + \tfrac{\partial \alpha_1(z)}{\partial z}(f_0(z, \xi_1) \\ &+ g_0(z, \xi_1, \xi_2, \epsilon)) \\ &- \tfrac{\partial V(z)}{\partial z} h_0(z, \alpha_1(z), \xi_1 - \alpha_1(z)) \end{aligned} \tag{8}$$

Then, there exists an $\epsilon^* > 0$ such that for all $0 < \epsilon < \epsilon^*$ the following hold *i) The fixed point equation*

$$\xi_2 = \psi_2(z, \xi_1, \xi_2, \epsilon), \quad (z, \xi_1) \in \mathcal{K}_z \times \mathcal{K}_{\xi_1}$$

has a unique smooth solution $\xi_2^* = \alpha_2(z, \xi_1, \epsilon)$.

ii) The state feedback $\xi_2 = \alpha_2(z, \xi_1, \epsilon)$ *semiglobally asymptotically stabilizes the origin for the* (z, ξ_1)*-subsystem of (7).*

iii) There exists a state feedback $u = \alpha_3(z, \xi_1, \xi_2, \epsilon)$ *that semiglobally asymptotically stabilizes the origin for the whole system in (7).*

Remark 3.2. It might seem that there are restricted classes of nonlinear systems in the form (7). But in fact, any nonlinear system $\dot{x} = f(x, u, \dot{u})$ that has a controllable linearization $\dot{x} = Ax + Bu$ around the origin can be transformed into (7) using a linear change of coordinates combined with a change of scale and time scale. We provide examples for this procedure later on.

Remark 3.3. Compared to theorem 3.1, the benefit of theorem 3.2 is that it is self-content and does not require any extra assumptions. On the other hand, theorem 3.2 requires explicit knowledge of a Lyapunov function $V(z)$ with $\dot{V} < 0$ $(\forall z \neq 0)$ in the definition of ψ_2 (equation (8)) that determines the fixed point controller $\alpha_2(z, \xi_1)$. Later, we resolve this problem in theorem 3.3.

Remark 3.4. In the definition of ψ_2 in theorem 3.2, the sigmoidal function $\sigma(s)$ is not restricted to be bounded.

Proof. The proof of (i) is simple and follows directly from contraction mapping theorem. To prove (ii), define the change of coordinates $\mu_1 = \xi_1 - \alpha_1(z)$ and consider the candidate Lyapunov function $W(z, \mu_1) = V(z) + \frac{1}{2}\mu_1^2$. Calculating $\dot{W}$ along the solution of the closed loop system, we get

$$\begin{aligned}
\dot{W} &= \frac{\partial V(z)}{\partial z} f_0(z, \alpha_1(z)) + \frac{\partial V(z)}{\partial z} g_0(z, \xi_1, \xi_2, \epsilon) \\
&+ \mu_1(\xi_2 - \frac{\partial \alpha_1(z)}{\partial z} f_0(z, \xi_1) - \frac{\partial \alpha_1(z)}{\partial z} g_0(z, \xi_1, \xi_2, \epsilon) \\
&+ \frac{\partial V(z)}{\partial z} h_0(z, \alpha_1(z), \mu_1)) \\
&= \frac{\partial V(z)}{\partial z} f_0(z, \alpha_1(z)) + \frac{\partial V(z)}{\partial z} g_0(z, \xi_1, \xi_2, \epsilon) \\
&- \mu_1 \sigma(\mu_1)
\end{aligned}$$

around the origin $(z, \mu_1) = (0, 0)$, the z-subsystem is locally exponentially stable given $\mu_1 = 0$. Because g_0 is at least quadratic in (z, ξ_1, ξ_2) and vanishes at $\epsilon = 0$, using a quadratic Lyapunov function the second term in the last equation is at least O(3) and therefore the origin is locally exponentially stable for the closed loop system on some small open neighborhood of the origin U_0 (that does not shrink by making $\epsilon > 0$ smaller). For any compact set of initial conditions $\mathcal{K}_0 := \mathcal{K}_z \times \mathcal{K}_{\mu_1} \ni (z(0), \mu_1(0))$ that contains the origin, take $\Omega_0 \supset \mathcal{K}_z$ as an invariant set of $V(z)$ (i.e. $\Omega_0 = \{z | V(z) \leq c_0, c_0 > 0\}$). Thus, $\tilde{\Omega}_0 := \Omega_0 \backslash U_0$ is a compact set. Take

$$\epsilon_0 = \min_{z \in \tilde{\Omega}_0} -\frac{\partial V(z)}{\partial z} f_0(z, \alpha_1(z)) > 0$$

then there exists an $\epsilon^* > 0$ such that for all $0 < \epsilon < \epsilon^*$, $|\frac{\partial V(z)}{\partial z} g_0| < \epsilon_0$ and $\dot{W} < 0$ over $\tilde{\Omega}_0$. Thus, any solution of the system enters U_0 after some finite time and then exponentially converges to the origin. The proof of (iii) is very straightforward and relays on the following elementary lemma and the result follows. □

Lemma 3.1. *(semiglobal backstepping) Consider the following system*

$$\begin{aligned}
\dot{z} &= f(z, \xi) \\
\dot{\xi} &= u
\end{aligned}$$

where f is smooth with $f(0, 0) = 0$. Suppose there exists a control law $\xi = \alpha(z)$ that asymptotically stabilizes $z = 0$ with a region of attraction $A \subset \Omega$ where Ω is an invariant set of a smooth positive definite proper Lyapunov function $V(z)$ with $\dot{V} < 0$ over $\Omega \backslash \{0\}$. Then, for a sufficiently large $c > 0$

$$u = -c(\xi - \alpha_1(z)) + \frac{\partial \alpha(z)}{\partial z} f(z, \xi)$$

semiglobally asymptotically stabilizes $(z, \xi) = (0, 0)$ for the composite system.

Now, we present a modified version of theorem 3.2 that does not require any explicit knowledge of a Lyapunov function $V(z)$ for the derivation of the control law.

Theorem 3.3. *(Fixed point backstepping III) Consider the nonlinear system in (7) and assume all the conditions in theorem 3.2 hold. Define*

$$\psi_2(z, \xi_1, \xi_2, \epsilon) = -\sigma(\xi_1 - \alpha_1(z)) + \frac{\partial \alpha_1(z)}{\partial z} f(z, \xi_1, \xi_2, \epsilon)$$

where $f = f_0(z, \xi_1) + g_0(z, \xi_1, \xi_2, \epsilon)$. Let $V(z)$ be a smooth positive definite proper Lyapunov function with $\frac{\partial V(z)}{\partial z} f_0(z, \alpha_1(z)) < 0$, $\forall z \neq 0$. In addition, assume there exists $c_0, r_0 > 0$ such that

$$|\frac{\partial V(z)}{\partial z} h_0(z, \alpha_1(z), \delta(t))| \leq c_0 V(z), \quad \forall |z| > r_0, \forall t > 0$$

where $\delta(t)$ is a smooth bounded function with an exponentially decaying tail (i.e. $\exists t_0, k_0, \gamma_0 > 0 : \forall t > t_0, |\delta(t)| \leq k_0 \exp(-\gamma_0(t - t_0))$). Then, all the results of theorem 3.2 hold.

Proof. Following the line of the proof in theorem 3.2, one can prove that the solution of the system is uniformly bounded for initial conditions in a compact set and then show that $\dot{W}$ will be negative after some finite time over an appropriate invariant set of $W(z, \mu_1)$ (The details are omitted). □

4 Examples

In this section, we present two examples of underactuated mechanical systems and apply our main theoretical results to control design for these systems.

4.1 The Cart-Pole System

The Cart-Pole system consists of a cart and an inverted pendulum on it (see Figure 1). The task is to stabilize the pole in its upright position at the origin from any initial condition in the upper half plane (i.e. $q_2 \in (-\pi/2, \pi/2)$). The Lagrangian equations of motion for the cart-pole system are

$$\begin{aligned}(m_1 + m_2)\ddot{q}_1 + m_2 l \cos(q_2)\ddot{q}_2 &= m_2 l \sin(q_2)\dot{q}_2^2 + F,\\ \cos(q_2)\ddot{q}_1 + l\ddot{q}_2 &= g\sin(q_2),\end{aligned} \tag{9}$$

where m_1 and q_1, are mass and position of the cart and m_2, l_2, and q_2 are mass, length of the link, and angle of the pole, respectively. Noting that $\cos(q_2) \neq 0$

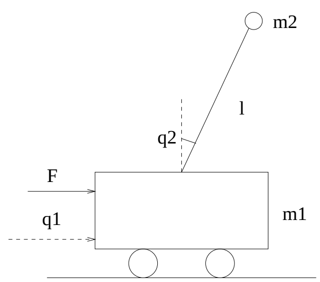

Figure 1: The Cart-Pole System

after taking $\ddot{q}_2 = u$ and canceling $\ddot{q}_1$ from the last two equations, by applying the following feedback law

$$\begin{aligned}F &= (m_2 l \cos(q_2) - (m_1 + m_2)l/\cos(q_2))u\\ &+ (m_1 + m_2)g\tan(q_2) - m_2 l \sin(q_2)\dot{q}_2^2\end{aligned}$$

the dynamics of the system can be written as the following (assuming $g = l = 1$)

$$\begin{aligned}\dot{q}_1 &= p_1,\\ \dot{p}_1 &= \tan(q_2) - u/\cos(q_2),\\ \dot{q}_2 &= p_2,\\ \dot{p}_2 &= u,\end{aligned}$$

using theorem 2.1, we can decouple the dynamics of the (q_1, p_1)-subsystem and the (q_2, p_2)-subsystem by applying the following global change of coordinates (q_2, p_2 are unchanged)

$$\begin{aligned}x_1 &= q_1 + \gamma(q_2),\\ x_2 &= p_1 + p_2/\cos(q_2),\end{aligned}$$

where

$$\gamma(q_2) = \int_0^{q_2} \frac{1}{\cos(s)} ds = \log\left(\frac{1 + \tan(q_2/2)}{1 - \tan(q_2/2)}\right), \tag{10}$$

in new coordinates the dynamics of the system transforms into the normal form (4) as the following

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= \tan(q_2)(1 + p_2^2/\cos(q_2)),\\ \dot{q}_2 &= p_2\\ \dot{p}_2 &= u\end{aligned} \tag{11}$$

To remove the constraint $|q_2| < pi/2$, we apply a second (global) change of coordinates and control as

$$\begin{aligned}\zeta_1 &= \tan(q_2),\\ \zeta_2 &= (1 + \tan(q_2)^2)p_2,\\ v &= (1 + \tan(q_2)^2)u + 2\tan(q_2)(1 + \tan(q_2)^2)p_2^2\end{aligned} \tag{12}$$

that transforms the dynamics of the system into

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= \zeta_1 + \zeta_1\zeta_2^2/(1 + \zeta_1^2)^{\frac{3}{2}},\\ \dot{\zeta}_1 &= \zeta_2,\\ \dot{\zeta}_2 &= v,\end{aligned} \tag{13}$$

which is a partially linear cascade nonlinear system in *strict feedforward form.*

Proposition 4.1. *There exists a smooth state feedback law determined by a fixed point controller that semiglobally asymptotically stabilizes the origin for the cart-pole system above the horizontal axis.*

Proof. First, apply the change of coordinates (or scale) and time scale as $z_1 = \epsilon^2 x_1, z_2 = \epsilon x_2, \xi_1 = \zeta_1, \xi_2 = \epsilon^{-1}\zeta_2, \tau = \epsilon t$. In new coordinates and time scale, the dynamics of the (x, ζ_1)-subsystem can be written as

$$\begin{aligned}\dot{z}_1 &= z_2,\\ \dot{z}_2 &= \xi_1 + \epsilon^2\omega(\xi_1)\xi_2^2,\\ \dot{\xi}_1 &= \xi_2,\end{aligned} \tag{14}$$

where $\omega(\xi_1) = \xi_1/(1 + \xi_1^2)^{\frac{3}{2}}$ is uniformly bounded in ξ_1 with $|\omega(\xi_1)| \leq \omega_{max} := \omega(1/\sqrt{2}) = 2/3\sqrt{3}$. Clearly, this system is in the form (7). After setting $\xi_2 = 0$, the z-subsystem is a double integrator with control ξ_1 that can be globally asymptotically stabilized using a linear state feedback $\xi_1 = \alpha_1(z) = -(z_1 + z_2)$ (or in general $\xi_1 = -\sigma(z_1 + z_2)$, where σ is a sigmoidal function). Based on theorem 3.3, there exists a fixed point controller $\xi_2 = \alpha_2(z, \xi_1, \epsilon)$ that for a sufficiently small ϵ semiglobally asymptotically stabilizes $(z, \xi_1) = (0, 0)$ for (14). This gives a semiglobally stabilizing state feedback $\zeta_2 = \tilde{\alpha}_2(x, \zeta_1, \epsilon)$ in the original coordinates and time scale for the (x, ζ_1)-subsystem in (13). Now, from lemma 3.1, a semiglobally stabilizing state feedback for the composite system in (13) can be obtained. □

Figure 2, shows the simulation results for the cart-pole system starting at the initial state $(1, 1, \pi/3, 0)$ for the choice of $\xi_1 = -\tanh(z_1 + z_2)$.

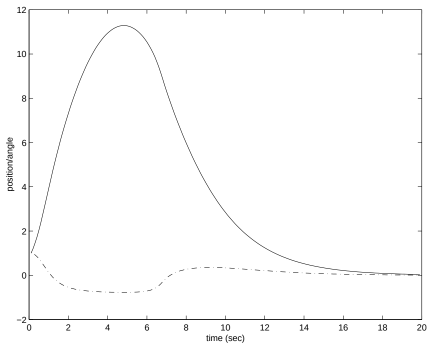

Figure 2: The state trajectory for the Cart-Pole System

Proposition 4.2. *There exists a globally stabilizing state feedback law for the cart-pole system in the form of nested small saturations.*

Proof. This simply follows from the fact that the dynamics of the cart-pole system is in strict feedforward form and satisfies all the conditions of Teel's nested saturations method in [10]. □

Remark 4.1. Though the controllers obtained by the method of nested saturations have global regions of attraction, applying relatively small control inputs leads to a very slow speed of convergence that might not be appropriate for practical purposes. The same is true for any low-gain design including the fixed point controllers with small parameters.

4.2 The Rotating Pendulum

The rotating pendulum is a system consisting of an inverted pendulum connected to an arm rotating within a horizontal plane (See Figure 3). The rotating pendulum is a rather complex underactuated mechanical system and many researchers have been interested in studying this system. The main available results on rotating pendulum is related to the swing up of the pendulum [11]. The Lagrangian equations of the motion for the rotating pendulum are in the form

$$M(q)\ddot{q} + C(q,\dot{q}) + G(q) = \begin{bmatrix} \tau \\ 0 \end{bmatrix}$$

where the elements of M, C, G matrices are given as the following:

$$\begin{aligned}
m_{11} &= J_1 + m_2 L_1^2 + m_2 l_2^2 \sin^2(q_2), \\
m_{21} &= m_{12} = m_2 L_1 l_2 \cos(q_2), \\
m_{22} &= J_2 + m_2 l_2^2, \\
c_1 &= 2m_2 l_2^2 \sin(q_2)\cos(q_2)\dot{q}_1\dot{q}_2 - m_2 L_1 l_2 \sin(q_2)\dot{q}_2^2, \\
c_2 &= -m_2 l_2^2 \sin(q_2)\cos(q_2)\dot{q}_1^2, \\
g_1 &= 0, \\
g_2 &= -m_2 g l_1 \sin(q_2), \\
\Delta(q) &= \det(M) = m_{11}m_{22} - m_{12}^2,
\end{aligned}$$

and m_i, l_i, L_i, J_i are masses, lengths of center of masses, lengths of the links, and inertia of the links, respectively. The task is to stabilize the pendulum in its upright position at the origin from any initial condition in the upper half plane. After applying the following

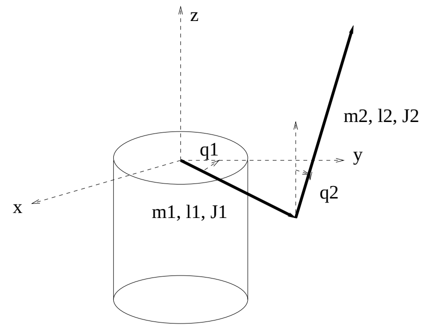

Figure 3: The Rotating Pendulum

feedback law

$$\tau(q,\dot{q}) = -\frac{\Delta}{m_{12}}u - \frac{m_{11}}{m_{12}}c_2 + c_1 - \frac{m_{11}}{m_{12}}g_2,$$

the dynamics of the rotating pendulum can be written as the following

$$\begin{aligned}
\dot{q}_1 &= p_1, \\
\dot{p}_1 &= k_2 \tan(q_2) + k_3 \sin(q_2) p_1^2 - (k_1/\cos(q_2))u, \\
\dot{q}_2 &= p_2, \\
\dot{p}_2 &= u,
\end{aligned}$$

where

$$\begin{aligned}
k_1 &= (J_2 + m_2 l_2^2)/m_2 L_1 l_2, \\
k_2 &= g l_1 / L_1 l_2, \\
k_3 &= l_2 / L_1.
\end{aligned}$$

using decoupling theorem (2.1), after a global change of the coordinates given by

$$\begin{aligned}
x_1 &= q_1 + k_1 \gamma(q_2), \\
x_2 &= p_1 + (k_1/\cos(q_2)) p_2,
\end{aligned}$$

where $\gamma(\cdot)$ is defined in (10), the dynamics of the system can be written as

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= k_2 \tan(q_2) + k_3 \sin(q_2)(x_2 - \tfrac{k_1}{\cos(q_2)} p_2)^2 \\
&+ \tan(q_2)\tfrac{k_1}{\cos(q_2)} p_2^2 \\
\dot{q}_2 &= p_2, \\
\dot{p}_2 &= u,
\end{aligned}$$

To remove the effect of the constraint $|q_2| < \frac{\pi}{2}$, we apply another change of coordinates and control as in (12).

The dynamics of the system transforms into

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= \zeta_1(k_2 + \frac{k_3}{(1+\zeta_1^2)^{\frac{1}{2}}}(x_2 - \frac{k_1}{(1+\zeta_1^2)^{\frac{1}{2}}}\zeta_2)^2\\ &+ \frac{k_1}{(1+\zeta_1^2)^{\frac{3}{2}}}\zeta_2^2),\\ \dot{\zeta}_1 &= \zeta_2,\\ \dot{\zeta}_2 &= v,\end{aligned}$$

that (unlike the cart-pole system) is *not in feedforward form.* In addition, the equation of $\dot{x}_2$ contains a $x_2\zeta_2$ term that can be problematic after a change of scale and time-scale (that depend on ϵ). To avoid this problem, one can apply the following change of coordinates

$$y_1 = x_1; y_2 = x_2\lambda(\zeta_1)$$

where

$$\lambda(\zeta_1) = (1+\zeta_1^2)^{k_4}; \quad k_4 := k_1k_3$$

and for all ζ_1, $\lambda(\zeta_1) \geq 1$. In new coordinates, we have

$$\begin{aligned}\dot{y}_1 &= y_2\frac{1}{\lambda(\zeta_1)},\\ \dot{y}_2 &= \zeta_1\lambda(\zeta_1)(k_2 + \frac{k_3}{\lambda^2(\zeta_1)(1+\zeta_1^2)^{\frac{1}{2}}}y_2^2 + \frac{k_1(k_4+1)}{(1+\zeta_1^2)^{\frac{3}{2}}}\zeta_2^2),\\ \dot{\zeta}_1 &= \zeta_2,\\ \dot{\zeta}_2 &= v,\end{aligned}$$

Proposition 4.3. *There exists a smooth static state feedback law determined by a fixed point controller that semiglobally asymptotically stabilizes the origin for the rotating pendulum above the horizontal plane.*

Proof. After a change of coordinates and time scale

$$z_1 = \epsilon^2 y_1, z_2 = \epsilon y_2, \xi_1 = \zeta_1, x_2 = \epsilon^{-1}\zeta_2, \tau = \epsilon t$$

the (y_1, y_2, ζ_1)-subsystem transforms into

$$\begin{aligned}\dot{z}_1 &= z_2\frac{1}{\lambda(\zeta_1)},\\ \dot{z}_2 &= \xi_1\lambda(\xi_1)(k_2 + \frac{k_3}{\epsilon^2\lambda^2(\xi_1)(1+\xi_1^2)^{\frac{1}{2}}}z_2^2\\ &+ \epsilon^2\frac{k_1(k_4+1)}{(1+\xi_1^2)^{\frac{3}{2}}}\xi_2^2),\\ \dot{\xi}_1 &= \xi_2,\end{aligned}$$

In lemma 4.1, it is shown that setting $\xi_2 = 0$ in the last equation, the z-subsystem can be globally stabilized using a linear feedback (or a saturated linear feedback). This means that the dynamics of the rotating pendulum satisfies all the conditions of theorem 3.3 and assuming that the condition in h_0 holds a semiglobally stabilizing feedback law exists for the rotating pendulum. □

Figure 4 shows the simulation results for the rotating pendulum with initial condition $(\pi/3, 0, \pi/4, 0)$.

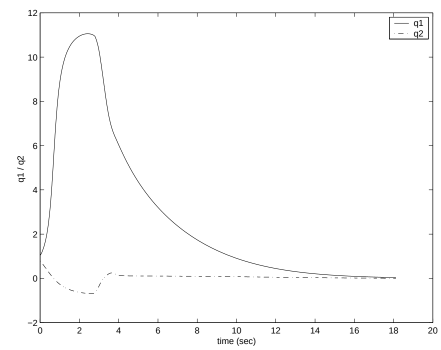

Figure 4: The state trajectory for the Rotating Pendulum

Lemma 4.1. *Consider the following second-order nonlinear system*

$$\begin{aligned}\dot{z}_1 &= z_2\frac{1}{\lambda(\zeta_1)},\\ \dot{z}_2 &= \xi_1\lambda(\xi_1)(k_2 + \frac{k_3}{\epsilon^2\lambda^2(\xi_1)(1+\xi_1^2)^{\frac{1}{2}}}z_2^2),\end{aligned} \tag{15}$$

the state feedback $\xi_1 = -(z_1+z_2)$ (or $\xi_1 = -\sigma(z_1+z_2)$) globally asymptotically stabilizes $z = 0$ for the closed loop system.

Proof. The proof is based on contraction of a Poincaré map from the positive z_2-axis to the negative z_2-axis using multiple Lyapunov functions (but so far no explicit Lyapunov function is known for this system). First, multiply the vector field of the system (15) by $\lambda(\xi_1) \geq 1$. Without loss of generality assume $k_2 = 1$. We get

$$\begin{aligned}\dot{z}_1 &= z_2,\\ \dot{z}_2 &= \xi_1\theta(z_2, \xi_1),\end{aligned} \tag{16}$$

where $\theta(z_2, \xi_1) = \lambda^2(\xi_1) + k_3z_2^2/\epsilon^2(1+\xi_1^2)^{\frac{1}{2}} \geq 1$. Now, we just need to prove given $\xi_1 = \alpha_1(z) = -(z_1+z_2)$, $z = 0$ is GAS for (16). Note that all the solutions of the closed loop system (16) are symmetric w.r.t. $z = 0$. Denote $A_1 = \{z|z_1 \geq 0, z_2 > 0\}$, $A_2 = \{z|z_1 \geq 0, z_2 \leq 0, z_1+z_2 \geq 0\}$, and $A_3 = \{z|z_1 \geq 0, z_2 < 0, z_1+z_2 \leq 0\}$ and note that $A_1 \cup A_2 \cup A_3$ is equal to the closed right half plane. Consider a solution of (16) starting at $P_1 = (0, a_1)$ with $a_1 > 0$. Because over A_1, $z_2(t)$ is strictly decreasing, after some finite time $t_1 > 0$, the solution intersects the z_1-axis at $P_2 = (a_2, 0)$, $a_2 > 0$. Given $V_1(z) = z_1^2 + z_2^2$, over A_1, $\dot{V}_1 < 0$, or $V_1(t_1) < V_1(0)$. That means $a_2 < a_1$. Similarly, for $V_2 = z_1^2 + (z_1+z_2)^2$ over A_2, $\dot{V}_2 \leq 0$, thus the solution intersects the line $z_1 + z_2 =: l(z) = 0$ after some finite time (because $l(z)$ is strictly decreasing) at $P_3 = (a_3, -a_3)$ with $0 \leq a_3 \leq a_2$. Now, we show that over A_3, both $z_1(t)$ and $z_2(t)$ decay exponentially. First, note that any solution of (16) starting at P_3 at $t = 0$ cannot intersect the line $l(z) = 0$ as long as the solution lies in A_3. Because the direction of the vector field on $l(z) = 0$ points out towards the inside of A_3. Thus for all $t \geq 0$, $z_1(t) + z_2(t) \leq 0$ over A_3 (the equality only holds at $t = 0$). Multiplying both sides of the last inequality by $z_1(t) > 0$, we have $\frac{d}{dt}(z_1^2) \leq -2z_1^2(t)$. Using comparison principle, we get $z_1(t) \leq z_1(0)e^{-t} = e^{-t}a_3$ over A_3. Also, because $\theta \geq 1$, $\dot{z}_2(t) = \xi_1\theta \geq -z_1(t) - z_2(t)$, or $\dot{z}_2(t) + z_2(t) \geq -z_1(t) \geq -z_1(0)e^{-t}$. Thus $\frac{d}{dt}(-z_2(t)e^t) \leq z_1(0)$, or $z_2(t) \geq -z_1(0)(t+1)e^{-t}$. This proves that both z_1 and z_2 decay exponentially over A_3. Also, by integrating $z_2(t)$, we get $z_1(t) \geq z_1(0)[(t+2)e^{-t} - 1]$. Therefore, for $0 \leq t \leq t^*$ where t^* is the solution of $(t+2)e^{-t} - 1 = 0$, $z_1(t) \geq 0$ and the solution wont leave A_3. A simple estimate shows that $t^* > \log(3) > 1$. Because over A_3, $z_1(t)$ is strictly decreasing, the solution intersects the negative z_2-axis at $P_4 = (-a_4, 0)$ where $a_4 \leq a_3(1+t^*)e^{-t^*} < \frac{2.1}{3}a_3$. The total contraction factor for the Poincaré map between P_1 and P_4 is less than $(1+t^*)e^{-t^*} < 1$ and thus the map is contractive. But any solution of (16) starting anywhere in the plane intersects the z_2-axis after some finite time. Therefore, all the solutions of (16) asymptotically converge to the origin and $z = 0$ is GAS. The proof for the bounded control is similar but more complicated and is omitted due to the space constraint. □

5 Conclusions

We considered stabilization of a special class of cascade nonlinear systems consisting of a nonlinear subsystem in cascade with a double integrator system. Despite the fact that these systems do not possess any triangular strict feedback structure, we showed how a backstepping type procedure applied to these systems naturally leads to a fixed point equation in the control input. We called this method fixed point backstepping and presented different types of fixed point backstepping procedures for global and semiglobal stabilization of this special class of cascade nonlinear systems. We showed how this method reduces the complexity of control design for a higher order system to control design for only the nonlinear lower order part of the system. We demonstrated this reduction strategy by applying our theoretical results to stabilization of the cart-pole system and the rotating pendulum to a point equilibrium over the upper half plane. For both of these systems semiglobal stabilization was achieved using fixed point controllers.

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