

Nonlinear Control and Reduction of Underactuated Systems with Symmetry I: Actuated Shape Variables Case

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Abstract

In this paper, we address nonlinear control and reduction of a class of high-order underactuated mechanical systems with kinetic symmetry called Class-I systems. Class-I systems are underactuated systems with actuated shape variables, decoupled inputs, and integrable normalized momentums (all to be defined). We show that all Class-I underactuated systems can be transformed into cascade systems in strict feedback form with a zero-dynamics that is a Lagrangian system itself.

1 Introduction

Control of underactuated mechanical systems has attracted many researchers in recent years. This is due to the broad range of real-life applications of underactuated systems in Robotics (e.g. mobile robots, flexible-link robots, snake-type robots, walking robots), Aerospace Vehicles (e.g. aircraft, spacecraft, helicopters, and satellites), and Marine Vehicles (e.g. surface vessels and underwater vehicles). See [4, 7] for recent surveys.

This paper is part I of a series of articles that aim to address reduction and nonlinear control of broad classes of high-order underactuated systems. In [4], it is shown that underactuated systems can be essentially classified to eight main classes that overall cover the majority of the aforementioned real-life applications. In part I, we focus on control of the first class of underactuated systems from the classification table of underactuated systems in [4] (p. 104).

The key in reduction of high-order underactuated systems is the kinetic symmetry properties of mechanical systems. Such a symmetry exists in most of real-life examples of underactuated systems. By kinetic symmetry, we mean the invariance of the kinetic energy under symmetry group actions. In the presence of the potential energy, this notion of symmetry is substantially different from the classical symmetry in [2].

The main contribution of part I of this paper is in providing change of coordinates in closed-form that transform high-order Class-I underactuated systems into cascade systems in strict feedback form. In addition, we show that the zero-dynamics corresponding to the

shape variables as the output is a lower-order Lagrangian system. In other words, the nonlinear part of the obtained normal form is physically meaningful.

The outline of the paper is as follows. In section 2, we provide some background on dynamics and symmetry properties of underactuated systems. In section 3, we present our main reduction and stabilization results. In section 4, an example is given. Finally, we make concluding remarks.

2 Underactuated Systems with Symmetry

In this paper, we consider the class of simple Lagrangian systems with configuration vector $q = \text{col}(q_x, q_s) \in Q_x \times Q_s$, a configuration space $Q = Q_x \times Q_s$ of dimension n , and the Lagrangian function

$$\mathcal{L}(q, \dot{q}) = K - V = \frac{1}{2} \dot{q}^T M(q_s) \dot{q} - V(q_x, q_s) \quad (1)$$

where K is the kinetic energy, $V(q)$ is the potential energy, and $M(q) = M(q_s)$ is the *inertia matrix*. We say the system has *kinetic symmetry* w.r.t. q_x due to

$$\frac{\partial K}{\partial q_x} = 0$$

We refer to q_x and q_s as the vectors of *external variables* and *shape variables*, respectively. The forced Euler-Lagrange equation for a system with Lagrangian (1) can be written as

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_x} - \frac{\partial \mathcal{L}}{\partial q} &= F_x(q) \tau \\ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_s} - \frac{\partial \mathcal{L}}{\partial q} &= F_s(q) \tau \end{aligned} \quad (2)$$

where $\tau \in \mathbb{R}^m$ and $F(q) = \text{col}[F_x(q), F_s(q)]$ is the *force matrix*. We say the mechanical system in (2) is an *underactuated system* if $m = \text{rank}(F(q)) < n$. Since $F(q)$

has full column rank, there exists a permutation of q such that $F(q)$ can be partitioned as $\text{col}[F_1(q), F_2(q)]$ where $F_2(q)$ is an invertible $m \times m$ matrix. If $F_1(q) \equiv 0$, we say (2) is a mechanical system with *decoupled inputs*. Otherwise, (2) is referred to as a mechanical system with *input coupling*. This paper is devoted to reduction and nonlinear control of high-order underactuated systems with fully-actuated shape variables and decoupled inputs. Without loss of generality, set $F_x(q) = 0, F_s(q) = I_m$ (otherwise, replace τ by $F_s^{-1}(q)\tau$).

By *reduction*, we mean transformation of the mechanical control system in (2) into a cascade nonlinear system in the form

$$\begin{aligned} \dot{x} &= f(x, \xi) \\ \dot{\xi} &= g(\xi, u) \end{aligned} \quad (3)$$

using a smooth invertible change of coordinates and control (i.e. diffeomorphism)

$$(x, \xi) = \Phi_1(q, \dot{q}), u = \Phi_2(q, \dot{q}, \tau) \quad (4)$$

The reduction of underactuated systems with partially-actuated/unactuated shape variables and input coupling are addressed in parts II and III of this paper in [6].

Remark 1. Notice that in general a system with Lagrangian (1) does not possess symmetry properties in the classical sense according to [2], unless $V(q_x, q_s) = V(q_s)$. As a result the generalized momentum $p_x = \partial \mathcal{L} / \partial \dot{q}_x$ conjugate to q_x is not anymore a conserved quantity for the unforced Lagrangian system. This is a fundamental difference between kinetic symmetry and classical symmetry.

The Lagrangian equations of motion in (2) with $F(q) = [0, I_m]^T$ can be rewritten as

$$\begin{aligned} m_{xx}(q_s) \ddot{q}_x + m_{xs}(q_s) \ddot{q}_s + h_x(q, \dot{q}) &= 0 \\ m_{sx}(q_s) \ddot{q}_x + m_{ss}(q_s) \ddot{q}_s + h_s(q, \dot{q}) &= \tau \end{aligned} \quad (5)$$

where h_x, h_s contain the Coriolis, centrifugal, and gravity terms. Due to Spong [10], the dynamics of an underactuated system in the form (5) can be partially linearized using an invertible change of control in the form $\tau = \alpha(q)u + \beta(q, \dot{q})$. This feedback is called a *collocated partially-linearizing feedback* [9] and it reduces the dynamics of q_s to a multi-variable double-integrator $\ddot{q}_s = u$.

The key tools in reduction of high-order underactuated systems with kinetic symmetry are generalized momentums, normalized momentums, and their integrals which are defined in the sequel. Let p_x be the *generalized momentum* conjugate to q_x , then for system (5)

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{q}_x} = m_{xx}(q_s)\dot{q}_x + m_{xs}(q_s)\dot{q}_s \quad (6)$$

We define the *normalized momentum* conjugate to q_x as

$$\pi_x = m_{xx}^{-1}(q_s)\frac{\partial \mathcal{L}}{\partial \dot{q}_x} = \dot{q}_x + m_{xx}^{-1}(q_s)m_{xs}(q_s)\dot{q}_s \quad (7)$$

We say the normalized momentum π_x is *integrable*, if there exists a smooth function $h(q) = (h_1(q), \dots, h_d(q))^T$ with $d = n - m$ such that $\pi_x = \dot{h}$ where $\dot{h} := Dh(q)\dot{q}$ and $Dh(q) = (\nabla h_1(q), \dots, \nabla h_d(q))$. Whenever π_x is integrable, we call $h(q)$ the *integral* of π_x .

Definition 1. (Class-I) We refer to the class of underactuated mechanical systems with actuated shape variables, decoupled inputs, and integrable normalized momentum π_x as *Class-I* systems.

3 Main Results

The following assumption is critical in reduction of Class-I underactuated systems (see definition 1).

Assumption 1. (integrability condition) Assume all the elements of

$$\omega = m_{xx}^{-1}(q_s)m_{xs}(q_s)dq_s \quad (8)$$

are exact one-forms and let $\omega = d\gamma(q_s)$. In other words, denoting $\mu(q_s) := m_{xx}^{-1}(q_s)m_{xs}(q_s)$, assume all the one-forms

$$\omega_i = \sum_{j=1}^m \mu_{ij}(q_s)dq_s^j$$

are exact for $i = 1, \dots, n - m$ (μ_{ij} and q_s^j are elements of $\mu(q_s)$ and q_s , respectively). Let $\omega_i = d\gamma_i(q_s)$, then $\gamma(q_s) := (\gamma_1(q_s), \dots, \gamma_m(q_s))^T$.

Here is our first main result on reduction of Class-I underactuated systems:

Theorem 1. Consider the underactuated system in (5) and suppose Assumption (1) holds. Then, the following global change of coordinates (i.e. diffeomorphism) obtained from the Lagrangian of the system

$$\begin{aligned} q_r &= q_x + \gamma(q_s) \\ p_r &= m_{xx}(q_s)\dot{q}_x + m_{xs}(q_s)\dot{q}_s = \frac{\partial \mathcal{L}}{\partial \dot{q}_x} \end{aligned} \quad (9)$$

transforms the dynamics of the underactuated system in (5) into a strict feedback cascade form

$$\begin{aligned} \dot{q}_r &= m_r^{-1}(q_s)p_r \\ \dot{p}_r &= g_r(q_r, q_s) \\ \dot{q}_s &= p_s \\ \dot{p}_s &= u \end{aligned} \quad (10)$$

where

$$m_r(q_s) := m_{xx}(q_s) \quad (11)$$

$$V_r(q_r, q_s) := V(q_r - \gamma(q_s), q_s) \quad (12)$$

$$g_r(q_r, q_s) := -\partial V_r(q_r, q_s)/\partial q_r \quad (13)$$

In addition, the (q_r, p_r) -subsystem is a Lagrangian system with configuration vector q_r

and reduced parameterized Lagrangian

$$\mathcal{L}_r(q_r, \dot{q}_r, q_s) = \frac{1}{2} \dot{q}_r^T m_r(q_s) \dot{q}_r - V_r(q_r, q_s) \quad (14)$$

that satisfies the unforced Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}_r}{\partial \dot{q}_r} - \frac{\partial \mathcal{L}_r}{\partial q_r} = 0 \quad (15)$$

Proof. By definition of q_r, p_r , we have

$$\dot{q}_r = m_{xx}^{-1}(q_s) p_r = m_r^{-1}(q_s) p_r$$

From the first line of the Euler-Lagrange equation in (2), we get

$$\dot{p}_r = \frac{\partial \mathcal{L}}{\partial q_x} = \frac{\partial K}{\partial q_x} - \frac{\partial V(q_x, q_s)}{\partial q_x} =: g_x(q_x, q_r)$$

due to kinetic symmetry w.r.t. q_x , $\partial K / \partial q_x = 0$. But

$$\begin{aligned} g_x(q_x, q_s) &= - \frac{\partial V(q_r - \gamma(q_s), q_s)}{\partial q_r} \frac{\partial q_r}{\partial q_x} \\ &= g_r(q_r, q_s) \times I_{n-m} = g_r(q_r, q_s) \end{aligned}$$

and the second line of (10) follows. Also, we have

$$p_r = \frac{\partial \mathcal{L}_r}{\partial \dot{q}_r}, \dot{p}_r = \frac{\partial \mathcal{L}_r}{\partial q_r}$$

which proves the reduced Lagrangian \mathcal{L}_r satisfies the unforced Euler-Lagrange equation. \square

Due to the strict feedback structure of the normal form of Class-I underactuated systems in (10), stabilization of the overall system reduces to stabilization of the (q_r, p_r) -subsystem with control input q_s . Global asymptotic stabilization of the (q_r, p_r) -subsystem is addressed in Theorem 2 (where the subscript “r” is dropped for simplicity). Using backstepping procedure, the controller for the composite system can be recursively obtained [1].

Theorem 2. Consider the following nonlinear system non-affine in control

$$\begin{aligned} \dot{q} &= N(u)p \\ \dot{p} &= g(q, u) \end{aligned} \quad (16)$$

where $q, p, u \in \mathbb{R}^n$, $g(q, u) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a smooth function with $g(0, 0) = 0$, $N(u)$ is an invertible matrix for all u , and $M(u) = N^{-1}(u)$ is a positive definite and symmetric inertia matrix. Suppose there exists an isolated root $u = \alpha(q)$ of $g(q, u) = 0$ with the property $\alpha(0) = 0$ such that

$$\det \left(\frac{\partial g}{\partial u}(q, \alpha(q)) \right) \neq 0$$

Set

$$\psi(q, v) := [M(u)g(q, u)]_{u=\alpha(q)+v}$$

Then, for all $q \in \mathbb{R}^n$, $w = \psi(q, v)$ is a local diffeomorphism around a neighborhood of $v = 0$. Assume there exists an open ball $B_r(0)$ around $w = 0$ and a function $\beta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$\psi(q, \beta(q, w)) = w, \quad \forall w \in B_r(0) \subset \mathbb{R}^n$$

uniformly in q . Let $\vec{\sigma}(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ denote the vector sigmoidal function

$$\vec{\sigma}(x) = \frac{x}{(1 + \|x\|^2)^{\frac{1}{2}}}$$

Then, for all $c_1, c_2 \in (0, r/2]$, the static state feedback $u = K(q, p)$ defined as the following

$$\begin{aligned} u &= K(q, p) := \alpha(q) + \beta(q, w), \\ w &= K_b(q, p) := -c_1 \vec{\sigma}(q) - c_2 \vec{\sigma}(p); \end{aligned} \quad (17)$$

globally asymptotically stabilizes the origin $(q, p) = (0, 0)$ for the nonlinear system in (16).

Proof. See pages 113–115 in [4]. \square

The following theorem gives a physical property of a special type of Class-I underactuated systems.

Theorem 3. *Consider the underactuated system in (5) and suppose in addition to Assumption (1) the following conditions hold:*

i) m_{xx} is constant.

ii) $V(q_x, q_s) = \frac{1}{2}q_x^T K q_x + U(q_s)$ where K is a positive definite matrix.

Then, the reduced (q_r, p_r) -subsystem in (10) is a general mass-spring system with configuration vector q_r and reduced Lagrangian

$$\mathcal{L}_r(q_r, \dot{q}_r) = \frac{1}{2}\dot{q}_r^T m_r \dot{q}_r - \frac{1}{2}q_r^T K q_r \quad (18)$$

satisfying the following forced Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}_r}{\partial \dot{q}_r} - \frac{\partial \mathcal{L}_r}{\partial q_r} = K \gamma(q_s) \quad (19)$$

with a nonlinear shape-dependent input force $K \gamma(q_s)$.

Proof. From the definition of $V(q_x, q_s)$ in ii), we get

$$g_r(q_r, q_s) = [-K q_x]_{q_x=q_r-\gamma(q_s)} = -K q_r + K \gamma(q_s)$$

and due to $m_r \ddot{q} = g_r(q_r, q_s)$ the result follows. \square

In [4], it is shown that the TORA system satisfies all the conditions of Theorem 3 and can be globally asymptotically stabilized using an explicit state feedback obtained from backstepping procedure.

4 Example

In this section, we provide an example of a Class-I underactuated system with four

degrees of freedom and two control inputs. Other examples of Class-I systems include the Acrobot [3], the TORA system [4, 11], and the Inertia Wheel Pendulum [5].

Example 1. (4 DOF, 2 Controls) Consider a Lagrangian system Σ with configuration vector $q = (x_1, x_2, \theta_1, \theta_2) \in \mathbb{R}^4$ and

$$\begin{aligned} M(q) &= \begin{bmatrix} I_{2 \times 2} & m_{12}(\theta) \\ m_{12}(\theta)^T & I_{2 \times 2} \end{bmatrix} \\ m_{12}(\theta) &= \begin{bmatrix} a \cos \theta_1 \cos \theta_2 & -a \sin \theta_1 \sin \theta_2 \\ 0 & a \cos \theta_2 \end{bmatrix} \\ V(q) &= \frac{1}{2}k_1 x_1^2 + \frac{1}{2}k_2 x_2^2 + \cos \theta_1 \cos \theta_2 \\ F(q) &= [0, I_{2 \times 2}]^T \end{aligned}$$

where $\theta = (\theta_1, \theta_2)^T$ and $k_1, k_2 > 0$ are constants. For $a \in (0, \frac{1}{4})$, $M(q)$ is a positive definite matrix. Due to $M(q) = M(\theta)$, θ_1 and θ_2 are two actuated shape variables of the system. Observe that both of the following one-forms

$$\begin{aligned} \omega_1 &= a \cos \theta_1 \cos \theta_2 d\theta_1 - a \sin \theta_1 \sin \theta_2 d\theta_2 \\ \omega_2 &= a \cos \theta_2 d\theta_2 \end{aligned}$$

are exact. Defining

$$\begin{aligned} \gamma_1(\theta) &= a \sin \theta_1 \cos \theta_2 \\ \gamma_2(\theta) &= a \sin \theta_2 \end{aligned} \quad (20)$$

we have $\omega_i = d\gamma_i(\theta)$, $i = 1, 2$. Set $\gamma(\theta) := (\gamma_1(\theta), \gamma_2(\theta))^T$ and define the reduced configuration q_r as $q_r = x + \gamma(\theta)$. Denoting $q_r = (z_1, z_2)^T$ and $p_r = \dot{q}_r = (w_1, w_2)^T$, in new coordinates we obtain the following cascade normal form for Σ

$$\begin{aligned} \Sigma_1 : & \begin{cases} \dot{z}_1 = w_1 \\ \dot{w}_1 = -k_1 z_1 + k_1 a \sin \theta_1 + \Delta(\theta) \\ \dot{\theta}_1 = v_1 \\ \dot{v}_1 = u_1 \end{cases} \\ \Sigma_2 : & \begin{cases} \dot{z}_2 = w_2 \\ \dot{w}_2 = -k_2 z_2 + k_2 a \sin \theta_2 \\ \dot{\theta}_2 = v_2 \\ \dot{v}_2 = u_2 \end{cases} \end{aligned} \quad (21)$$

where the perturbation term $\Delta(\theta)$ is defined as $\Delta(\theta) := k_1 a \sin \theta_1 (\cos \theta_2 - 1)$. Notice that $\Delta(\theta)$ vanishes at $\theta_2 = 0$ uniformly in θ_1 . In addition, due to $\|\Delta(\theta)\| \leq 2k_1 a$, $\Delta(\theta)$ is uniformly bounded in θ . In fact, the unperturbed system in (21) with $\Delta = 0$ consists of the normal forms of two independent TORA systems Σ_1, Σ_2 (see [4], pages 127–131). Let $u_i = K_i(z_i, w_i, \theta_i, v_i), i = 1, 2$ be the globally asymptotically stabilizing state feedback laws for the 2 d.o.f. TORA system. After an elementary Lyapunov-based argument, it can be shown that any solution of the Σ_1 -subsystem of the closed-loop system in (21) is bounded. Based on Sontag’s theorem on stability of cascade systems [8], the origin for the overall closed-loop system in (21) is globally asymptotically stable.

5 Conclusion

In this paper, we addressed reduction and stabilization of high-order underactuated systems with actuated shape variables, decoupled inputs, and integrable normalized momentums (i.e. Class-I systems). We proved that all Class-I underactuated systems can be transformed into cascade nonlinear systems in strict feedback form. In addition, we showed that the nonlinear part of the obtained normal form is itself a Lagrangian system. Furthermore, we introduced a globally asymptotically stabilizing state feedback for this nonlinear part. We applied our results to global stabilization of a Class-I underactuated system with 4 DOF and 2 controls.

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