

Control of Underactuated Mechanical Systems with two Degrees of Freedom and Symmetry

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Abstract

In this paper, we consider a special class of underactuated mechanical systems with two degrees of freedom and symmetry. By symmetry, we mean the inertia matrix of the system is independent of the unactuated degree of freedom. We show that there exists a natural global change of coordinates obtained from the Lagrangian of the system that transforms the system into a partially linear cascade nonlinear system that is strict feedback. The nonlinear part of this system is non-affine in control and this highly complicates control design for the system. We provide conditions under which this nonlinear subsystem can be globally stabilized and give globally stabilizing control laws for it. The strict feedback structure of the system in new coordinates allows us to obtain a globally stabilizing control law for the composite system using standard backstepping. We apply our result to global asymptotic stabilization of the Acrobot.

1 Introduction

Throughout this paper, we consider an underactuated mechanical system with two degrees of freedom and symmetry. By symmetry, we mean the inertia matrix of the system is independent of the unactuated degree of freedom. In other words, denoting the configuration vector of the system by $q = (q_1, q_2)$ where q_1 and q_2 denote the unactuated and actuated degrees of freedom, respectively. We say the system has *configuration symmetry* if the inertia matrix $M(q)$ is independent of q_1 . Influenced by the procedure of Lagrangian reduction, the symmetric properties of the system allows us to obtain a natural change of coordinates from the Lagrangian of the system that globally transforms the dynamics of the system to a partially linear cascade nonlinear system in strict feedback form (see definition 2.2). Now, if a globally stabilizing control law is available for the nonlinear part, using standard backstepping procedure one can obtain a globally stabilizing control law for the composite system. In this paper, we

give the conditions such that this global stabilization is possible for the aforementioned class of underactuated mechanical system. In addition, we give globally stabilizing control laws in analytically explicit form for this class of underactuated systems. We apply our results to global stabilization of the Acrobot that is a two-link revolute planar robot with an actuator at the elbow. In [3], it is shown that rather general underactuated mechanical system can be at least partially linearized by linearization of the actuated part using a change of control input. But the new control appears in the both actuated and unactuated parts of the original system and this highly complicates control design for these systems. We show that how Lagrangian reduction avoids this and allows us to decouple the actuated and unactuated parts. This decoupling method was first introduced by the authors for the special case of the Acrobot in [2]. Here our main result is to introduce a new control design method for the class of underactuated systems with 2 d.o.f. and symmetry. Moreover, we generalize our results in [2] for more general class of underactuated mechanical systems.

Here is an outline of the paper. We start with some basic definitions. Then state our main results. Next, as an example we consider the Acrobot. Finally, we give concluding remarks.

2 Basic Definitions

Definition 2.1. By a *sigmoidal function* $\sigma(s)$, we mean a smooth function that is bounded, strictly increasing, and has the following properties: i) $\sigma(0) = 0$, ii) $s\sigma(s) > 0, \forall s \in \mathbb{R} \setminus \{0\}$.

Definition 2.2. ([1]) We say a system is in *strict feedback form* if it has the following triangular structure

$$\begin{aligned}\dot{x} &= f(x, \xi_1), \\ \dot{\xi}_1 &= \xi_2, \\ &\dots \\ \dot{\xi}_m &= v,\end{aligned}$$

3 Main Results

Consider an underactuated mechanical system with two degrees of freedom and configuration variable $q = (q_1, q_2)$ where q_1 and q_2 denote the unactuated and actuated configuration variables, respectively. The Lagrangian of this system can be written as

$$L(q, \dot{q}) = \frac{1}{2} \dot{q}^T \begin{bmatrix} m_{11}(q_2) & m_{12}(q_2) \\ m_{21}(q_2) & m_{22}(q_2) \end{bmatrix} \dot{q} - V(q) \quad (3.1)$$

where $m_{ij}(q_2)$ are the elements of the inertia matrix $M(q_2)$ and $V(q)$ is the potential energy of the system. The Euler-Lagrange equation for this mechanical system is as the following

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \begin{bmatrix} 0 \\ \tau \end{bmatrix}$$

where τ is the input applied to q_2 (apparently, q_1 is unactuated). The last equation in the vector form can be expressed as

$$\begin{bmatrix} \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_2} \end{bmatrix} - \begin{bmatrix} \frac{\partial L}{\partial q_1} \\ \frac{\partial L}{\partial q_2} \end{bmatrix} = \begin{bmatrix} 0 \\ \tau \end{bmatrix} \quad (3.2)$$

Denote

$$g_i(q_1, q_2) = -\frac{\partial V(q)}{\partial q_i}, \quad i = 1, 2$$

To show that the actuated part of the system can be linearized, define

$$s(q, \dot{q}) = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} - M(q_2) \ddot{q}$$

where $s = (s_1, s_2)^T$. Then, applying the following feedback law

$$\tau(q, \dot{q}) = \tilde{m}_{22} u + \tilde{s}_2$$

where

$$\begin{aligned} \tilde{m}_{22} &= m_{22} - m_{21} m_{11}^{-1} m_{12} \\ \tilde{s}_2 &= s_2 - m_{21} m_{11}^{-1} s_1 \end{aligned}$$

transforms the second equation in (3.2) into $\ddot{q}_2 = u$ that is a double integrator as

$$\begin{aligned} \dot{q}_2 &= p_2 \\ \dot{p}_2 &= u \end{aligned}$$

This change of control is invertible because $\det(M(q_2)) \neq 0$. Noting that

$$\frac{\partial L}{\partial q_1} = -\frac{\partial V(q)}{\partial q_1} = g_1(q_1, q_2),$$

from the first line of equation (3.2) immediately it follows that

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} = g_1(q_1, q_2)$$

This is the key to our first main result:

Theorem 3.1. *Consider an underactuated system with two degrees of freedom and symmetry w.r.t the unactuated configuration variable. There exists a (natural) global change of coordinates obtained from the Lagrangian of the system as*

$$\begin{aligned} z_1 &= q_1 + \gamma(q_2) \\ z_2 &= m_{11}(q_2) \dot{q}_1 + m_{12}(q_2) \dot{q}_2 = \frac{\partial L}{\partial \dot{q}_1} \end{aligned} \quad (3.3)$$

where q_2, p_2 are unchanged and

$$\gamma(q_2) = \int_0^{q_2} \frac{m_{12}(\theta)}{m_{11}(\theta)} d\theta \quad (3.4)$$

that transforms the dynamics of the system into a strict feedback form as

$$\begin{aligned} \dot{z}_1 &= z_2 / m_{11}(q_2) \\ \dot{z}_2 &= g_1(z_1 - \gamma(q_2), q_2) \\ \dot{q}_2 &= p_2 \\ \dot{p}_2 &= u \end{aligned} \quad (3.5)$$

Proof. Taking $z_2 = \frac{\partial L}{\partial \dot{q}_1}$, from the first line of (3.2) immediately it follows that

$$\dot{z}_2 = g_1(q_1, q_2)$$

but

$$\frac{z_2}{m_{11}(q_2)} = \dot{q}_1 + \frac{m_{12}(q_2)}{m_{11}(q_2)} \dot{q}_2$$

because the last term is integrable, taking $z_1 = q_1 + \gamma(q_2)$, we have $\dot{z}_1 = z_2 / m_{11}(q_2)$ and substituting $q_1 = z_1 - \gamma(q_2)$ in $g_1(q_1, q_2)$ finishes the proof. \square

The system in (3.5) is in a special form that under certain conditions allows us to design a globally stabilizing control law for the z -subsystem. Then, using backstepping procedure, a globally stabilizing control law for the composite system can be obtained [1]. Specially, we use the sign definite property of $m_{11}(q_2) > 0$. Here is our second main result that eventually gives the stabilizing control law for (3.5):

Theorem 3.2. *Consider the following system*

$$\begin{cases} \dot{q} = p, \\ \dot{p} = f(p) + g(q, u), \end{cases} \quad (3.6)$$

where f and g are C^1 functions with $f(0) = 0$ and $g(0, 0) = 0$, f is decreasing ($f' \leq 0$), and $q, p, u \in \mathbb{R}$. Suppose that zero is not a critical value for $g(q, u)$ and

$$\frac{\partial g(q, u)}{\partial u} \neq 0$$

on the manifold

$$M = \text{Ker}(g) = \{(q, u) \in \mathbb{R}^2 : g(q, u) = 0\}$$

Then, $g(q, u)$ has an isolated root $\alpha(q)$ such that

$$g(q, \alpha(q)) = 0$$

and there exists a continuously differentiable state feedback law in the following form

$$u = \alpha(q) - \sigma(q + p), \quad (3.7)$$

that globally asymptotically stabilizes the origin for (3.6) (σ is a sigmoidal function).

Proof. See Appendix A for the proof. \square

Remark 3.1. Equation 3.6 can be viewed as the dynamics of a mechanical system with one degree of freedom q , a nonlinear control input $g(q, u)$, and dissipation terms $f(p)$.

The following proposition shows that how theorem 3.2 can be applied to control design for (3.5) and provides all the conditions under which the original underactuated mechanical system with symmetry can be globally stabilized. Here is the combination of our main results.

Proposition 3.1. Define $\bar{g}_1(z_1, q_2) = g_1(z_1 - \gamma(q_2), q_2)$ and assume \bar{g}_1 satisfies all the conditions of g in theorem 3.2. Let $\alpha(z_1)$ be the isolated root of $\bar{g}_1(z_1, q_2) = 0$. Then, there exists a feedback law in the form

$$q_2 = k(z_1, z_2) := \alpha(z_1) - \sigma(z_1 + z_2)$$

that globally asymptotically stabilizes the origin $z = 0$ for the z -subsystem of (3.5) as

$$\begin{aligned} \dot{z}_1 &= z_2 / m_{11}(q_2) \\ \dot{z}_2 &= \bar{g}_1(z_1, q_2) \end{aligned} \quad (3.8)$$

In addition, there exists a globally asymptotically stabilizing control law for the composite system in (3.5) (based on backstepping procedure).

Proof. Multiplying the vector field in (3.8) by $m_{11}(q_2)$, we get

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= m_{11}(q_2) \bar{g}_1(z_1, q_2) =: g(z_1, q_2) \end{aligned} \quad (3.9)$$

because $m_{11}(q_2) > 0$ for all q_2 , a control law $q_2 = k(z)$ that globally stabilizes the origin $z = 0$ for (3.9) also globally stabilizes $z = 0$ for (3.8). the roots of $g(z_1, q_2) = 0$ given by $M = \text{Ker}(g) = \{(z_1, q_2) | g(z_1, q_2) = 0\}$ are the same as the roots of $\bar{g}_1(z_1, q_2)$. On the other hand, calculating $\frac{\partial g}{\partial q_2}$ over M , one gets

$$\begin{aligned} \frac{\partial g}{\partial q_2} &= \frac{dm_{22}(q_2)}{dq_2} \bar{g}_1(z_1, q_2) + m_{22}(q_2) \frac{\partial \bar{g}_1}{\partial q_2} \\ &= m_{22}(q_2) \frac{\partial \bar{g}_1}{\partial q_2} \neq 0 \end{aligned}$$

that means $g(z_1, q_2)$ satisfies all the conditions of theorem 3.2 and the result follows. \square

4 Example

In this section, we apply our result to global stabilization of the Acrobot around its upright position. The Acrobot is a two-link revolute planar robot with one actuator at the elbow as shown in Figure 1. The iner-

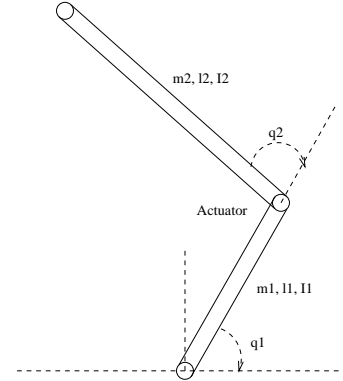


Figure 1: The Acrobot

tia matrix for the Acrobot is given by

$$\begin{aligned} m_{11}(q_2) &= m_1 l_1^2 + m_2 (L_1^2 + l_2^2 + 2L_1 l_2 \cos(q_2)) + I_1 + I_2 \\ m_{12}(q_2) &= m_{21}(q_2) = m_2 (l_2^2 + L_1 l_2 \cos(q_2)) + I_2 \\ m_{22}(q_2) &= m_2 l_2^2 + I_2 \end{aligned}$$

where q_i , m_i , L_i , l_i , and I_i denote angles, masses, lengths, lengths of center of masses, and inertia, respectively. Apparently, the inertia matrix of the Acrobot is independent of q_1 . The potential energy for the Acrobot is as

$$V(q_1, q_2) = (m_1 l_1 + m_2 L_1) g_0 \sin(q_1) + m_2 l_2 g_0 \sin(q_1 + q_2)$$

(g_0 is the gravity constant) and thus

$$\begin{aligned} g_1(q_1, q_2) &= -\frac{\partial V}{\partial q_1} = -(m_1 l_1 + m_2 L_1) g_0 \cos(q_1) \\ &\quad - m_2 l_2 g_0 \cos(q_1 + q_2) \end{aligned}$$

Following the line of proposition 3.1, we need to show that

$$\begin{aligned} \bar{g}_1(z_1, q_2) &= -(m_1 l_1 + m_2 L_1) g_0 \cos(z_1 - \gamma(q_2)) \\ &\quad - m_2 l_2 g_0 \cos(z_1 - \gamma(q_2) + q_2) \end{aligned}$$

satisfies all the conditions of g in theorem 3.2. For doing so, denote

$$A = (m_1 l_1 + m_2 L_1), \quad B = m_2 l_2, \quad \phi = z_1 - \gamma(q_2)$$

\bar{g}_1 can be rewritten as

$$\bar{g}_1(z_1, q_2) = -g_0(A \cos(\phi) + B \cos(\phi + q_2))$$

we need to prove if $\bar{g}_1 = 0$ then $\frac{\partial \bar{g}_1}{\partial q_2} \neq 0$. By contradiction suppose both $\bar{g}_1 = 0$ and $\frac{\partial \bar{g}_1}{\partial q_2} = 0$. Thus

$$A \cos(\phi) + B \cos(\phi + q_2) = 0$$

and noting $\frac{\partial \phi}{\partial q_2} = -\frac{m_{12}(q_2)}{m_{11}(q_2)}$, we get

$$A \frac{m_{12}}{m_{11}} \sin(\phi) - B \frac{m_{11} - m_{12}}{m_{11}} \sin(\phi + q_2) = 0$$

from the last two equations and the property $\cos^2(\phi + q_2) + \sin^2(\phi + q_2) = 1$ after some calculations we get

$$\sin^2(\phi) = \frac{1 - \frac{B^2}{A^2}}{m_{11}(m_{11} - 2m_{12})} (m_{11} - m_{12})^2$$

At this point, several sufficient conditions can be provided such that the last equation has no solutions in ϕ . The following condition is one possible sufficient condition that guarantees both $\bar{g}_1 = 0$ and $\frac{\partial \bar{g}_1}{\partial q_2}$ cannot vanish together.

Assumption 4.1. Assume that physical parameters of the Acrobot satisfy

$$i) m_1 l_1 + m_2 L_1 > m_2 l_2 \text{ (i.e. } A > B).$$

$$ii) I_1 + m_1 l_1^2 + m_2 L_1 > I_2 + m_2 l_2^2.$$

For example, if two links have uniform density and $l_1 \geq l_2$, this assumption holds. In other words, \bar{g}_1 satisfies all the conditions in theorem 3.2 and the globally stabilizing control law for the Acrobot can be obtained from proposition 3.1. To obtain the isolated root $q_2 = \alpha(z_1)$ of $\bar{g}_1(z_1, q_2) = 0$, differentiate the last equation w.r.t. a parameter s . We get

$$\frac{\partial \bar{g}_1}{\partial z_1} \frac{dz_1}{ds} + \frac{\partial \bar{g}_1}{\partial q_2} \frac{dq_2}{ds} = 0$$

or

$$\begin{aligned} \frac{dz_1}{ds} &= 1 \\ \frac{dq_2}{ds} &= -\frac{\partial \bar{g}_1}{\partial z_1} / \frac{\partial \bar{g}_1}{\partial q_2} \end{aligned}$$

the solution of this ODE with zero initial conditions both positive and negative in time gives a parameterization of the roots of $\bar{g}_1(z_1, q_2) = 0$ by z_1 as $M = (z_1, \alpha(z_1))$. Figures 2 and 3 show simulation results for the Acrobot.

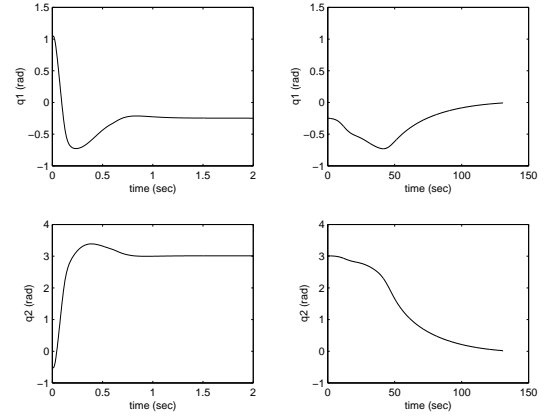


Figure 2: Trajectory of (q_1, q_2) for the Acrobot

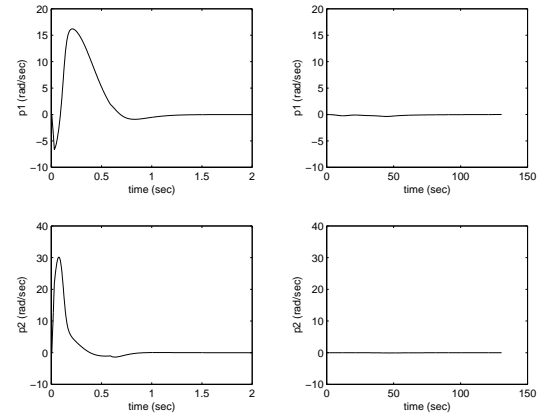


Figure 3: Trajectory of (p_1, p_2) for the Acrobot

5 Conclusions

We considered a class of underactuated mechanical system with symmetry and with respect to the unactuated degree of freedom. We proved that there exists a natural global change of coordinates obtained from the Lagrangian of the system that transforms the dynamics of the system into a partially linear nonlinear system in strict feedback form. The nonlinear part of the system in new coordinates is highly complex and non-affine in control. We introduced a control design method for global stabilization of the nonlinear part that is second-order system non-affine in control. We applied our results to global stabilization of the Acrobot.

Appendix

A Proof of theorem 3.2

Because zero is not a critical value for g , $M = g^{-1}(0) = \text{Ker}(g)$ is a one dimensional differentiable manifold in R^2 . In other words, M is a two-dimensional curve in

plane that divides R^2 into two connected regions:

$$A^+ = \{(q, u) \in R^2 : u > \alpha(q)\}$$

and

$$A^- = \{(q, u) \in R^2 : u < \alpha(q)\}.$$

But $g_u(q, u) \neq 0$ on M , based on implicit mapping theorem it follows that $g(q, u) = 0$ has an isolated root $u = \alpha(q)$ that is a C^1 function and M can be parameterized as

$$M = \{(q, \alpha(q)) : q \in R\}.$$

Also, on M either $g_u(q, u) > 0$ or $g_u(q, u) < 0$ (otherwise, at some point on M $g_u(q, u) = 0$). Without loss of generality assume

$$\frac{\partial g(q, u)}{\partial u} \Big|_{u = \alpha(q)} > 0$$

by continuity of $g_u(q, u)$ there exists a tubular neighborhood of M

$$N_\delta(M) = \{(q, u) \in R^2 : |u - \alpha(q)| < \delta, \delta > 0\}$$

such that $g_u(q, u) > 0$ on $N_\delta(M)$. Suppose $|\sigma(s)| < \delta, \forall s$ where σ is a sigmoidal function and $\delta > 0$. Then, given

$$u = k(q, p) = \alpha(q) - \sigma(q + p),$$

$(q, k(q, p)) \in N_\delta(M), \forall q, p$ and therefore

$$g_u(q, k(q, p)) > 0$$

for all q, p . Now, consider the closed loop system (3.6) with control input $u = k(q, p)$ as the following

$$\begin{cases} \dot{q} = p, \\ \dot{p} = f(p) + g(q, \alpha(q) - \sigma(q + p)), \end{cases} \quad (\text{A.1})$$

we prove that the origin $(q, p) = (0, 0)$ is globally asymptotically stable for (A.1). Define

$$h(q, p) = -f(p) - g(q, \alpha(q) - \sigma(q + p)),$$

and

$$H(q) = \int_0^q h(s, 0) ds$$

Noting that $sh(s, 0) > 0, \forall s \neq 0$, $H(q)$ is positive semidefinite with $H(0) = 0$ and $H(q) > 0, \forall q \neq 0$. Consider the following Lyapunov function candidate for (A.1)

$$V(q, p) = H(q) + \frac{1}{2}p^2,$$

we have

$$\dot{V} = p(h(q, 0) - h(q, p)),$$

but

$$h_p(q, p) = -f'(p) + \sigma'(q + p) \cdot g_u(q, k(q, p)) > 0,$$

therefore, $\dot{V} \leq 0, \forall q, p$. But in $\dot{V}(q, p) = 0$ or $\{(q, p) : p = 0\}$ the only invariant set is the origin $(q, p) = (0, 0)$. From LaSalle's invariance principle, it follows that the origin is globally asymptotically stable for (A.1). This finishes the proof.

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