Magnetoquasistatic Response of Conducting and Permeable Prolate Spheroid Under Axial Excitation

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Abstract—An analytical solution is presented for the problem of magnetic diffusion into and scattering from a permeable, highly but not perfectly conducting prolate spheroid under axial excitation, expressed in terms of an infinite matrix equation. The spheroid is assumed to be embedded in a homogeneous nonconducting medium as appropriate for low-frequency, high-contrast scattering governed by magnetoquasistatics. The solution is based on separation of variables and matching boundary conditions where the prolate spheroidal wavefunctions with complex wavenumber parameter are expanded in terms of spherical harmonics. For small skin depths, an approximate solution is developed that avoids any reference to the spheroidal wavefunctions. The problem of long spheroids and long circular cylinders is solved by using an infinite cylinder approximation. In some cases, our ability to evaluate the spheroidal wavefunctions breaks down at intermediate frequencies. To deal with this, a general broadband rational function approximation technique is developed and demonstrated. We treat special cases and provide numerical reference data for the induced magnetic dipole moment or, equivalently, the magnetic polarizability factor.

Index Terms—Conducting and permeable prolate spheroid, electromagnetic induction, magnetic diffusion, magnetoquasistatics.

I. INTRODUCTION

LOW-FREQUENCY electromagnetic induction methods are promising candidates for the development of advanced techniques for the detection and discrimination of subsurface objects such as unexploded ordnance (UXO) and landmines buried in soil [1]–[3]. Innovative broadband sensors are being engineered [3], [4]. In the frequency range under consideration here (from about 30 Hz to 300 kHz) rough air-ground interfaces and most soil dielectric heterogeneities have an insignificant influence, a clear advantage relative to ground penetrating radar (GPR) [5]. The conductivity of metallic targets exceeds that of soil by many orders of magnitude so that secondary sources on and within the targets dominate. Also, in this frequency range, displacement currents are negligible both within the target and its surroundings. These considerations motivate interest in solutions for the magnetoquasistatic response of conducting and permeable objects embedded in a homogeneous and insulating medium [6]. Note that in this idealized model, the fields in the background medium obey Laplace’s equation, while inside the target they are governed by magnetic diffusion, i.e., a Helmholtz equation with imaginary squared wavenumber. The frequency range of interest encompasses the scattering regimes from near magnetostatics up to the limiting case of vanishing skin depth, when the object can be replaced by an equivalent perfectly conducting boundary, still within the magnetoquasistatic regime.

Objects of general shape can be analyzed with the help of a surface integral formulation and the method of moments (MoM). Results for conducting and permeable bodies of revolution were reported in [7]. The problem of a prohibitively fine discretization for skin depths that are small compared to the largest target dimension can be addressed by a special formulation taking into account the exponential decay of the fields inside the object [8]. Other numerical techniques under development include the method of auxiliary sources (MAS) [9].

The recent investigative activity in this area has increased the demand for analytical solutions for canonically shaped objects needed for testing of numerical codes, calibration of instruments, and the development of model-based inversion methods. Theoretical investigations can also aid in improving the understanding of some relatively counterintuitive diffusion phenomena.

A benchmark solution is that for a conducting and permeable sphere, originally developed with focus on geophysical applications [10]–[14]. We consider here the problem of a conducting and permeable prolate spheroid (elongated ellipsoid of revolution) with the exciting uniform primary field along the major axis. The prolate spheroid is of fundamental interest because it includes the special cases of the sphere and, in the limit of infinite length, the circular cylinder. Furthermore, it is an example of an orientable object that exhibits a continuously varying surface curvature. The oblate spheroid (flattened ellipsoid of revolution) can be treated in a manner similar to what is presented in this paper. For a prolate spheroid, the response to axial excitation field components is typically the strongest.

The key ingredients of one of the solutions advanced here (based on scalar spheroidal wavefunctions) are mentioned in [14], [15] but integral equation approaches to the boundary value problem are favored there. In [16] no numerical results were reported and the solution given appears not to satisfy the boundary conditions at the surface of the spheroid. The lack of a comprehensive treatment of the spheroidal magnetic diffusion problem in the open literature is the chief motivation of the
present work which also aims at providing numerical reference data. The formally more complicated use of vector spheroidal wavefunctions for both axial and transverse (and, thus, oblique) primary fields is demonstrated in [17], from which our approach here is distinguished in that we rely on both the electric and magnetic fields to construct the ultimate magnetoquasistatic solution.

It should be noted that the problem of plane wave scattering from dielectric spheroids, the solution of which is highly developed [18]–[28], is quite different from the magnetic diffusion or electromagnetic induction problem considered here. Both phenomena are governed by a Helmholtz equation. However, the squared wavenumber inside a dielectric body is real, while inside a conducting and permeable object under magnetoquasistatic excitation it is imaginary. The elementary primary field in magnetic diffusion is not a plane wave but a spatially uniform, time-varying magnetic field. In the radar case, an important far-field quantity is the scattering cross-section while in electromagnetic induction we are interested in the induced magnetic dipole moment. Numerically, the exact solution of the diffusion problem requires the evaluation of the spheroidal wavefunctions with complex wavenumber parameter while routines in low-level computational languages such as Fortran are readily available only for the real case [29]. Similarities between the radar and the electromagnetic induction problems include the appearance of infinite systems of equations due to the nonorthogonality of the spheroidal wavefunctions for different wavenumbers. In principle, it might be possible to reduce the formal solution of the appropriate plane-wave scattering problem to that of the electromagnetic induction problem by considering the mathematical limit as the wavenumber of the exterior space approaches zero and treating the imminent singularities correctly. However, this statement is of no immediate practical value. We have found it more convenient to solve the electromagnetic induction problem directly under the tailored and simplifying assumptions of magnetoquasistatics.

In what follows, Section II presents the formal solution of the boundary value problem. It is specialized in Section II-A to a succinct expression for the induced magnetic dipole moment of the scattering body. In Section II-B we determine an asymptotic high-frequency form, which in the limit provides the correct normalization value. A low-frequency form is obtained in Section II-C, which provides a check and a simplified expression for the magnetic polarizability \( R_{\text{magn}} \) for high relative permeabilities or gives a limiting \( R_{\text{magn}} \) value under finite permeability but large elongation. In Section II-D, the same expression for \( R_{\text{magn}} \) is obtained for the general case, with high permeability as was obtained for the low-frequency case in Section II-C. The asymptotic high-frequency form of Section II-B is enhanced in Section III to provide an approximation for the general solution at high frequencies. An approximate solution for large elongations is developed in Section IV. Section V presents the numerical implementation of results. These reveal that evaluation of the spheroidal wavefunctions breaks down, rather suddenly, at and above intermediate frequencies. The problem is exacerbated for large permeabilities and elongations. However, using the aforementioned enhanced high-frequency approximation, we succeed in obtaining values at the high-frequency end of the spectrum, leaving possibly a gap only at intermediate frequencies. This gap can be bridged by the use of a broadband rational approximation, explained in Section VI. The conclusion ends the body of the paper, followed by an appendix deriving useful forms for expressing and calculating the magnetic dipole moment of an arbitrarily shaped scattering body.

The time convention adopted in the paper is \( e^{-i\omega t} \) and this factor is suppressed throughout.

II. Solution of Boundary Value Problem

We consider a prolate spheroid of conductivity \( \sigma \) and permeability \( \mu \) with major axis or length

\[
\ell = d \xi_0
\]

and minor axis

\[
2a = d \sqrt{\ell^2 - 1}
\]

as shown in Fig. 1. In other words, the interfocal distance of the spheroid is [30]

\[
d = \sqrt{\ell^2 - 4a^2}
\]

and \( \xi_0 \) is given by

\[
\xi_0 = \frac{\ell}{\sqrt{\ell^2 - 4a^2}}
\]

with \( d \to 0 \) and \( \xi_0 \to \infty \) in the limiting case of a sphere and \( d \to \ell \) and \( \xi_0 \to 1 \) in the limiting case of a long circular cylinder with needlelike end caps (acicular limit). The spheroid is centered about the origin of a prolate spheroidal coordinate system \( (\eta, \zeta, \phi) \) with both the interfocal distance and the rotational axis coinciding with those of the spheroid so that \( \zeta = \xi_0 \) describes the surface of the spheroid \( |\eta| \leq 1 \) and \( \zeta \geq 1 \). As explained earlier, the homogeneous background medium of permeability \( \mu \) is considered to be nonconducting.

A uniform primary magnetic field in the direction of the axis of rotation and alternating with angular frequency \( \omega \) is given by

\[
\vec{H}_0 = \hat{z} H_{0z} e^{i\omega t}.
\]

In view of the rotational symmetry of the configuration, the total electric field is of the form

\[
\vec{E} = \hat{\phi} E_\phi(\eta, \zeta)
\]

and thus, divergence-free. It is readily shown that in this case the vector wave equation for \( \vec{E} \) reduces to a scalar Helmholtz equation and can be reduced further to the scalar Helmholtz equation

\[
\nabla^2 (E_\phi e^{\pm ik} \phi) + k^2 E_\phi e^{\pm ik} = 0.
\]

The introduction of the auxiliary factor \( e^{\pm ik} \) in (7) is strictly a mathematical convenience and the product field \( E_\phi e^{\pm ik} \) has no direct physical meaning.

In view of (7), inside the spheroid (region 1), the electric field \( E_\phi = E_{k\phi} \) is expanded in terms of radial and angular spheroidal wave functions [30] with index \( m = 1 \) as

\[
E_{k\phi}(\eta, \zeta) = H_{0z} i\omega \mu_1 \frac{d}{2} \sum_{n=1}^{\infty} A_n R_{1n}^{(2)}(c_1, \zeta) S_{2n}(c_1, \eta)
\]
where the spheroidal wavenumber parameter

\[ c_1 = k_\ell \frac{d}{2} = \frac{d}{2} \sqrt{\omega \mu \sigma_1} \quad (9) \]

is complex with argument \( \pi/4 \) (provided that the product \( \omega \mu \sigma_1 \) is positive real which is the case in all the numerical examples considered). In region 2 outside the spheroid, \( k^2 = 0 \) so that (7) becomes Laplace’s equation, and we may express the electric field in terms of associated Legendre functions of the first and second kind as [31]

\[ E_{2\phi}(\eta, \xi) = H_0 \omega \mu \frac{d}{2} \]

\[ \times \left[ \frac{1}{2} \sqrt{\xi^2 - 1} P^1_1(\eta) + \sum_{n=1}^{\infty} B_n Q^1_n(\xi) P^2_n(\eta) \right] \quad (10) \]

where we follow Flammer [30] in defining \( P^m_n \) in terms of Legendre polynomials by

\[ P^m_n(\eta) = (1 - \eta^2)^{m/2} \frac{d^m}{d\eta^m} P_n(\eta) \quad (11) \]

which is different by a factor of \((-1)^m\) from the standard definition [32].

In (10), the infinite series represents the exterior secondary or scattered field that vanishes as \( \xi \to \infty \). The first term is induced by the uniform primary magnetic field in the absence of the scatterer and can also be written as

\[ E_{0\phi} = H_0 \omega \mu \frac{P}{2} \quad (12) \]

where we refer to cylindrical coordinates \((z, \rho, \phi)\). If the scattered field is made vanishingly weak, the electric field (6) reduces to

\[ \hat{E}_0 = \hat{\phi} E_{0\phi}(\rho, z). \quad (13) \]

By integrating Faraday’s law or, equivalently

\[ -\frac{\partial}{\partial z} E_{0\phi} = 0 \quad (14) \]

and

\[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho E_{0\phi} \right) = H_0 \omega \mu \quad (15) \]

we find that (13) with \( E_{0\phi} \) given by (12) is the only possible electric field \( E_0 \) produced by (5) and complying with (6). The presence of the spheroid imposes, via the boundary condition for the tangential electric field, a symmetry on the exterior electric field that otherwise would be determined only up to an additive irrotational field, due to the assumed absence of any currents in the exterior space.

The nonzero components of the total magnetic field follow from (8), (10) and Faraday’s law, applying the curl operator in prolate spheroidal coordinates. They are given, in region 1, by

\[ H_{1\eta}(\eta, \xi) = \frac{H_0}{\sqrt{\xi^2 - \eta^2}} \sum_{n=1}^{\infty} A_n T_{1n}(c_1, \xi) S_{1n}(c_1, \eta) \quad (16) \]

\[ H_{2\eta}(\eta, \xi) = -\frac{H_0}{\sqrt{\xi^2 - \eta^2}} \sum_{n=1}^{\infty} A_n R^{(1)}_{1n}(c_1, \xi) \]

\[ \times \frac{d}{d\eta} \left[ (1 - \eta^2) S_{1n}(c_1, \eta) \right] \quad (17) \]

with the derivative

\[ T_{mn}(c_1, \xi) = \frac{d}{d\xi} \left[ \sqrt{\xi^2 - 1} R^{(1)}_{mn}(c_1, \xi) \right] \quad (18) \]

and in region 2 by

\[ H_{2\eta}(\eta, \xi) = \frac{H_0}{\sqrt{\xi^2 - \eta^2}} \]

\[ \times \left[ \xi P^1_1(\eta) + \sum_{n=1}^{\infty} B_n U_{1n}(\xi) P^1_n(\eta) \right] \quad (19) \]

\[ H_{2\xi}(\eta, \xi) = -\frac{H_0}{\sqrt{\xi^2 - \eta^2}} \]

\[ \times \left[ \frac{1}{2} \sqrt{\xi^2 - 1} V_{11}(\eta) + \sum_{n=1}^{\infty} B_n Q^1_n(\xi) V_{1n}(\eta) \right] \quad (20) \]

where

\[ U_{mn}(\xi) = \frac{d}{d\xi} \left[ \sqrt{\xi^2 - 1} Q^m_n(\xi) \right] \quad (21) \]

\[ V_{mn}(\eta) = \frac{d}{d\eta} \left[ \sqrt{1 - \eta^2} P^m_n(\eta) \right]. \quad (22) \]

The boundary conditions of continuous tangential electric and magnetic field require, respectively

\[ \sum_{n=1}^{\infty} \left[ A_n T_{1n}(c_1, \xi) S_{1n}(c_1, \eta) \right. \]

\[ - B_n U_{1n}(\xi) P^1_n(\eta) = \frac{H_0}{2} \sqrt{\xi^2 - 1} P^1_1(\eta) \quad (23) \]

\[ \sum_{n=1}^{\infty} \left[ A_n T_{1n}(c_2, \xi) S_{1n}(c_1, \eta) \right. \]

\[ - B_n U_{1n}(\xi) P^1_n(\eta) = \xi_0 P^1_1(\eta). \quad (24) \]
In a crucial step, the angle functions are now expanded as
\[ S_{nm}(c_1, \eta) = \sum_{r=0}^{\infty} d_{n,m}^{m^r}(c_1) P_{n+r}^m(\eta) \]  
(25)
where the summation starts with \( r = 0 \) for \( n - m \) even and with \( r = 1 \) when \( n - m \) odd and the prime means that, beginning with the first, only every second term in the summation is kept. The numerical evaluation of the spheroidal expansion coefficients \( d_{n,m}^{m^r} \), which will also be utilized in the computation of the radial wavefunctions \( R_{n,m}^{(1)} \) [based on the expansion (39) below], is discussed in Section V. Inserting (25) in (23) and (24) and using the orthogonality of the associated Legendre functions [32]
\[ \int_{-1}^{1} d\eta' P_{m}^{2}(\eta') P_{n}(\eta) = \frac{2m(m+1)}{2m+1} \delta_{mn} \]  
(26)
where \( \delta_{mm} \) is the Kronecker delta, we find from multiplying by \( P_{n}^{1}(\eta) \) and integrating for \( m = 1, 2, \ldots \)
\[ \sum_{n=1}^{\infty} A_{2n-1} \mu R_{1,2n-1}^{(1)}(c_1, \xi_0) d_{2n+2}^{2n-1}(c_1) \]
\[ - B_{2n-1} \mu Q_{2n-1}^{1}(\xi_0) \]
\[ - B_{2n-1} U_{1,2n-1}(\xi_0) = \xi_0 \delta_{(2n-1)1} \]  
(27)
Testing with \( P_{2n}^{1}(\eta) \), on the other hand, shows that all harmonics with even index vanish, i.e., \( A_{2n} = B_{2n} = 0 \). This is expected from the symmetry of the problem, which implies \( E_{0}(\xi, \eta) = E_{0}(\xi, -\eta) \). Eliminating the \( B_{2n-1} \) from (27) and (28), we arrive at the infinite system of linear equations given by
\[ \sum_{n=1}^{\infty} A_{2n-1} \mu R_{1,2n-1}^{(1)}(c_1, \xi_0) d_{2n+2}^{2n-1}(c_1) \]
\[ - \mu Q_{2n-1}^{1}(\xi_0) \]
\[ = \mu \left[ \frac{1}{2} \sqrt{\xi_0^2 - 1} U_{1,2n-1}(\xi_0) \right] \]
\[ = \mu \left[ \frac{1}{2} \sqrt{\xi_0^2 - 1} \delta_{(2n-1)1} \right] \]  
(29)
After solving (29), the \( B_{2n-1} \) can be obtained directly from
\[ B_{2n-1} = \frac{1}{\mu Q_{2n-1}^{1}(\xi_0)} \]
\[ \times \left[ \mu \sum_{n=1}^{\infty} A_{2n-1} \mu R_{1,2n-1}^{(1)}(c_1, \xi_0) d_{2n+2}^{2n-1}(c_1) \right] \]
\[ - \mu \left[ \frac{1}{2} \sqrt{\xi_0^2 - 1} \delta_{(2n-1)1} \right] \]  
(30)
which concludes the formal solution of the boundary value problem. In the following, we discuss the induced magnetic dipole moment and various limiting cases of the theory. The numerical implementation of (29), (30) is described in Section V.

A. Induced Magnetic Dipole Moment

The secondary magnetic field due to an induced magnetic dipole of moment \( \mathbf{m} \) at \( \mathbf{r} = \mathbf{r}_0 \) is given by [1], [7], [33]
\[ \mathbf{B}_s(\mathbf{r}) = \frac{3\mathbf{r} \times \mathbf{r'}}{4\pi r'^3} \cdot \mathbf{m} \]  
(31)
where
\[ \mathbf{r} \] identity tensor;
\[ \mathbf{r'} \] unit vector pointing from point \( \mathbf{r}_0 \) to point \( \mathbf{r} \); 
\[ r' \] distance between the points.
Due to the rapid decay of the higher-order multipole fields, \( \mathbf{m} \) is a quantity of primary practical interest in electromagnetic induction methods. For our spheroid problem, the induced dipole moment is found from (19) and (20), where the secondary field is represented by the infinite series involving the coefficients \( B_{n} \). As \( \xi \to \infty \), the term for \( n = 1 \) dominates. By using the leading order terms of the expansions
\[ Q_{1}^{1}(\xi) = -\frac{2}{3} \xi^{-2} - \frac{7}{15} \xi^{-4} + O(\xi^{-6}) \]  
(32)
\[ U_{11}(\xi) = \frac{2}{5} \xi^{-2} + \frac{2}{5} \xi^{-4} + O(\xi^{-6}) \]  
(33)
and noting that, as \( \xi \to \infty \), the surfaces \( \xi = \text{const.} \) become spherical and [30]
\[ \frac{d}{d \xi} \xi \rightarrow r', \ \eta \rightarrow \cos \theta \]  
(34)
with the polarizability factor for prolate spheroids under axial excitation
\[ R_{1}^{||}(\xi_0) = -\frac{2Q_{1}^{1}(\xi_0)}{\sqrt{\xi_0^2 - 1}} B_1. \]  
(35)
As shown in the following, the normalization of \( R_{1}^{||}(\xi_0) \) is such that:
\[ R_{1}^{||}(\xi_0) \rightarrow 1 \]  
(37)
in the high-frequency limit. Because in (35), (36) frequency only enters implicitly through \( B_1 \), (35) is also a statement of the induced dipole moment in this limit. The value of the associated Legendre function \( Q_{1}^{1}(\xi_0) < 0 \) required in the normalization is given in terms of elementary functions by
\[ Q_{1}^{1}(\xi_0) = -\frac{1}{2} \sqrt{\xi_0^2 - 1} \left( \frac{2\xi_0}{\xi_0^2 - 1} - \ln \left( \frac{\xi_0 + 1}{\xi_0 - 1} \right) \right). \]  
(38)
B. High-Frequency Limit

The limit (37) is most conveniently verified by setting \( E_{2\phi} \) according to (10) to zero at \( \xi = \xi_0 \) (equivalence of perfectly conducting spheroid and spheroid with vanishing skin depth) and noting that all \( B_{n} \) other than \( B_1 \) are zero.

Obtaining the high-frequency limit directly from (29), (30) requires an investigation of the asymptotic behavior of the radial
spheroidal wavefunction and its derivative. Consider the expansion [30]

$$R^{(1)}_{mn}(c_1, \xi) = \left( \frac{\xi^2 - 1}{\xi^2} \right)^m \left[ \sum_{r=0}^{\infty} \frac{(2m+r)!}{r!} d^{(m)}_{r}(c_1) \right]^{-1} \times \sum_{r=0}^{\infty} r^{m-n} \frac{(2m+r)!}{r!} d^{(m)}_{r}(c_1) j_{m+r}(c_1 \xi)$$  (39)

where the summation convention is the same as for (25) and the $j_n$ are the spherical Bessel functions of the first kind [32]. As $\text{Im}(c_1) \to \infty$, the right-hand side of

$$j^{n+1}_{m+r}(c_1 \xi) \sim \frac{j^{n+1}_{m}(c_1 \xi)}{2 c_1 \xi}$$  (40)

is in fact independent of the summation index $r$. Thus (39) becomes asymptotically, for a finite $\xi$

$$R^{(1)}_{mn}(c_1, \xi) \sim \frac{j^{n+1}_{m}(c_1 \xi)}{2 c_1 \xi}$$  (41)

and we find, inserting (41) in the definition (18) and dividing by (41)

$$\frac{T{mn}(c_1, \xi)}{R^{(1)}_{mn}(c_1, \xi)} \sim -i c_1 \sqrt{\xi^2 - 1}.$$  (42)

With the help of (42), we now obtain from (29)

$$\sum_{n=1}^{\infty} A_{2n-1} R^{(1)}_{1(2n-1)}(c_1, \xi_0) d^{(2n-1)}_{2m-2}(c_1)$$

$$\sim \frac{\mu}{2} U_{1(2m-1)}(\xi_0) + i \mu c_1 \sqrt{\xi^2 - 1} Q^{2m-1}_{2m-1}(\xi_0)$$

indicating that as expected, the $B_n$ other than $B_1$ vanish to leading order and, with (44) from (36)

$$\frac{\mu}{2} U_{1(2m-1)}(\xi_0) + i \mu c_1 \sqrt{\xi^2 - 1} Q^{2m-1}_{2m-1}(\xi_0)$$

$$\sim \frac{\mu}{2} \left[ \frac{1}{2} \sqrt{\xi^2 - 1} - \xi_0 Q^{2m-1}_{2m-1}(\xi_0) \right] U_{1(2m-1)}(\xi_0) \delta_{(2m-1)1}$$  (53)

which, however, does not depend on frequency. Substituting (53) into (30), we find

$$B_{2m-1} \sim -\frac{\xi_0}{U_{1(2m-1)}(\xi_0)} \delta_{(2m-1)1}$$  (54)

As $c_1$ increases with frequency, the terms containing $\mu_1$ become negligible so that (45) reduces to (37). We will return to (45) in Section III.

C. Low-Frequency Limit

In the low-frequency limit we observe that, because the spheroidal angle functions $S_{mn}$ reduce to the associated Legendre functions of the first kind [30], or

$$d^{(m)}_{r}(c_1) \to \delta_{(n-m)r}.$$  (46)

in (25), the (29) for the $A_n$ decouple and only $A_1$ and $B_1$ in (30) are nonzero. Obtaining $B_1$ by eliminating $A_1$ and making use of the asymptotic relations

$$R^{(1)}_{11}(c_1, \xi_0) \sim c_1 \frac{j_{1}(c_1 \xi_0) \sqrt{\xi^2_0 - 1}}{c_1 \xi_0} \sim \frac{c_1}{3} \sqrt{\xi^2_0 - 1}$$  (47)

and thus

$$T_{11}(c_1, \xi_0) \sim \frac{2}{3} c_1 \xi_0$$  (48)

we find from (36)

$$\frac{R_{1\pro}^{(1)}}{\mu_1 \sqrt{\xi^2_0 - 1} U_{11}(\xi_0) - 2 \mu_1 \xi_0 Q^{1}_{1}(\xi_0)}$$  (49)

where

$$U_{11}(\xi_0) = \xi_0 \ln \frac{\xi_0 + 1}{\xi_0 - 1} - 2.$$  (50)

The limit (49) can be shown to be in agreement with the corresponding result in magnetostatics given, e.g., in [34]. If in addition to the low-frequency limit, we consider the acicular limit $\ell/2a \to \infty$, then (49) reduces to

$$\frac{R_{1\pro}^{(1)}}{-\frac{\mu - \mu_1}{\mu} = -\chi_m}$$  (51)

which is the negative magnetic susceptibility of the spheroid. If the elongation of the spheroid is fixed but the relative permeability goes to infinity, $\mu_1/\mu \to \infty$, (49) simplifies to

$$\frac{R_{1\pro}^{(1)}}{-2 \xi_0 Q^{2}_{1}(\xi_0) \sqrt{\xi^2_0 - 1} U_{11}(\xi_0).}$$  (52)

D. Limiting Case of Large Relative Permeability

Another interesting limiting case of (29), (30) is that of large relative permeability $\mu_1/\mu \to \infty$, but with $c_1$ fixed. More direct than in the high-frequency limit, (29) in this case yields again a closed-form expression for the infinite sum in (30), given by

$$\sum_{n=1}^{\infty} A_{2n-1} R^{(1)}_{1(2n-1)}(c_1, \xi_0) d^{(2n-1)}_{2m-2}(c_1)$$

$$\sim \frac{\mu}{2} U_{1(2m-1)}(\xi_0) + i \mu c_1 \sqrt{\xi^2 - 1} Q^{2m-1}_{2m-1}(\xi_0)$$

which, however, does not depend on frequency. Substituting (53) into (30), we find

$$B_{2m-1} \sim -\frac{\xi_0}{U_{1(2m-1)}(\xi_0)} \delta_{(2m-1)1}$$  (54)

and, using (54) in (36), we again arrive at the specialized low-frequency limit (52).

III. APPROXIMATE SOLUTION FOR HIGH FREQUENCIES

As we will see in Section V, while our frequency range of interest spans several orders of magnitude, the practical solution of (29), (30) is possible only for low to intermediate frequencies. The breakdown of this otherwise exact formulation,
which is found to be a sudden one, is due to the expansion of the spheroidal wavefunctions into spherical harmonics (25), (39), which is usable only for sufficiently small magnitude of \( c_1 \) [30]. It is therefore important to develop an alternative formulation tractable at high frequencies or to develop at least approximate or asymptotic solutions that, ideally, would have a range of validity that at the low-frequency end overlaps the highest frequencies at which the evaluation based on the expansion into spherical harmonics is still possible.

In Section II-B, we derived (45) using the leading-order high-frequency asymptotic behavior of the radial spheroidal wavefunction of the first kind and its derivative. We note that this formula still depends on \( c_1 \), which makes it a possible candidate for the purposes of this section. Attractive, beside the simple functional form of (45) is the fact that, as \( \mu_t/\mu \to \infty \) with \( c_1 \) fixed, (45) simplifies to (52) (see also Section II-D). Thus, for large relative permeability, a case of practical importance, (45) might in fact be a good broadband approximation since it approaches the correct asymptotic values at both the low-frequency and the high-frequency end. However, numerical tests showed that this is true only for the nearly spherical case \( \ell/2\alpha \approx 1 \). In general, the extremum of the imaginary part of (45) occurs at a frequency that is too high and, consequently, the real part fails to connect with the exact solution computable at low frequencies. This undesirable behavior makes the formula less useful in practice.

However, an alternative derivation of (45) paves our way to an approximation that does work. For this purpose, partially inspired by the starting point of the numerical approach for small skin depths advanced in [8], let us consider a special separable form of the electric field inside the spheroid for high frequencies

\[
E_{2\phi}(\eta, \xi) \approx E_{2\phi}(\eta, \xi_0) e^{\gamma_1(\xi - \xi_0)} \tag{55}
\]

where \( \gamma_1 \) is a dimensionless propagation constant yet to be determined. With (55), the boundary condition on the tangential electric field is satisfied automatically, by construction. Inserting (55) into (7), we quickly discover that if we are to satisfy the wave equation at least approximately, we have to set \( \gamma_1 = -i c_1 \), where the negative sign is chosen based on the physical reasoning that the field inside the spheroid propagates inward and decays exponentially away from the surface. Then, from Faraday’s law with the curl operator in prolate spheroidal coordinates, we can find the \( H_{1\eta} \) corresponding to (55). By matching with (19), satisfying the remaining boundary condition on the tangential magnetic field and using (10), we are led back to (44) and what follows there.

Thus, reconsidering (55), we conclude that this approximation contains a basic problem, namely, it ignores the variation of the radius of curvature along the surface of the spheroid (a problem that disappears in the spherical limit). Thus, if we are trying to implement the picture of a wave locally one-dimensional (1-D) in the coordinate normal to the boundary and with complex wavenumber \( k_1 \), we have to take into account the metric of the spheroidal coordinate system, motivating the expression

\[
E_{1\phi}(\eta, \xi) \approx E_{2\phi}(\eta, \xi_0) e^{-i k_1 \int_{\xi_0}^{\xi} \frac{d\xi'}{h_\xi(\eta, \xi')}}, \tag{56}
\]

with the metrical coefficient [30]

\[
h_\xi(\eta, \xi) = d \frac{\sqrt{\xi^2 - \eta^2}}{\xi^2 - 1}. \tag{57}
\]

Before proceeding, we note that near the tips or poles of the spheroid where \( |\eta| \approx 1 \), (55) and (56) in fact coincide [specialize (57) and recall (9)]. Because of the smaller radius of curvature there, this offers an explanation for the erroneous shift of the extremum of the imaginary part of (45) to higher frequencies. Furthermore, according to (4), in the spherical limit we have \( \xi_0 \to \infty \) and thus \( \xi \gg 1 \) in the thin surface layer where (56) again reduces to (55). Also, the original derivation leading to (45) used an asymptotic formula for the radial wavefunction only but not for the angular wavefunction or its expansion coefficients \( d_{nm} \). Neglecting the angular aspect of the spheroidal problem, which was convenient for deriving the high-frequency limit appears now to be related to the shortcomings of (45) as a high-frequency approximation.\(^3\)

Using (56) in Faraday’s law without further approximation yields

\[
H_{1n}(\eta, \xi) \approx \frac{2}{\kappa_0} \int \frac{\mu_1}{\mu} U_{2n}(\xi_0) \frac{\xi_0}{\sqrt{\xi_0^2 - 1}} Q_n(\xi_0) \left[ -i c_1 \sqrt{\xi_0^2 - \eta^2} Q_n(\xi_0) \right]^* P_n(\eta) + i c_1 \sqrt{\xi_0^2 - \eta^2} Q_n(\xi_0) \right] P_n^*(\eta), \tag{58}
\]

Matching the remaining boundary condition on the tangential magnetic field we obtain from (10), (19), (56) and (58)

\[
\sum_{n=1}^{\infty} B_n^m \left[ \frac{\mu_1}{\mu} U_{2n}(\xi_0) - \frac{\xi_0}{\sqrt{\xi_0^2 - 1}} Q_n(\xi_0) \right] P_n^m(\eta) = -\frac{\xi_0}{2\mu_1 - \mu} + \frac{i c_1}{2} \sqrt{\xi_0^2 - \eta^2} \right] P_n^m(\eta) \tag{59}
\]

to be enforced for all \( \eta \). Introducing the auxiliary function

\[
\Pi_{m,m}(\xi) = \frac{2m+1}{2m(m+1)} \int_{-1}^{1} d\eta \sqrt{\xi^2 - \eta^2} P_m^1(\eta) P_m^1(\eta), \tag{60}
\]

and using (26), we get from (59) by testing\(^5\) with \( P_m^1(\eta) \) for \( m = 1, 2, \ldots \)

\[
B_{2n-1}^m \left[ \frac{\mu_1}{\mu} U_{2n-1}(\xi_0) - \frac{\xi_0}{\sqrt{\xi_0^2 - 1}} Q_{2n-1}(\xi_0) \right] + i c_1 \sum_{n=1}^{\infty} B_{2n-1}^m Q_{2n-1}(\xi_0) \Pi_{2n-1}(2n-1) = -\frac{\xi_0}{2\mu_1 - \mu} + \frac{i c_1}{2} \sqrt{\xi_0^2 - 1} \Pi_{2n-1}(2n-1) \xi_0 \xi_1 \tag{61}
\]

\(^3\) Even though we have reviewed some of the literature on the asymptotics of the angular spheroidal wavefunction \([35]–[41]\), so far, we have not been led to any alternative formulation for the high-frequency case.

\(^5\) Since, according to the definition (60) and the parity of the associated Legendre functions, \( \Pi_{m,m}(\xi) = 0 \) whenever \( m - n \) odd, all the \( B_m \) with \( m \) even vanish as expected.
Once the $B_{3n}$ have been obtained from the infinite system of (61), the high-frequency approximation for the response in the far field represented by $R_{3n0}^{\parallel}$ follows from (36). We will follow this procedure numerically in Section V, where a straightforward Gauss-Legendre quadrature of (60) is employed. Finally, we observe that, as $\mu_2/\mu \rightarrow \infty$ with $c_1$ fixed, (61) again simplifies to (52), indicating that the approximation introduced here can be expected to extend to lower frequencies when the relative permeability is large.

IV. APPROXIMATE SOLUTION FOR LARGE ELONGATIONS

To obtain an approximation of the magnetic polarizability factor $R_{3n0}^{\parallel}$ of a long conducting and permeable prolate spheroid, we note that, as $\ell/2a$ becomes large, the spheroid fills the interior of a circular cylinder while its needlelike poles retreat to infinity. Thus, we can expect the responses of a long spheroid and a long circular cylinder to be similar.

For a circular cylinder with radius $a$, length $\ell$, permeability $\mu_1$ and conductivity $\sigma_1$, where $\ell \gg 2a$, we estimate the current distribution and magnetic field internal to the cylinder by those found for an infinitely long cylinder of the same radius, permeability, conductivity and orientation.\(^6\) We then obtain the approximation for the induced magnetic dipole moment of the finite-length cylinder based on the exact functional

\[
\tilde{m}_l [J(\tau), R(\tau)] = \frac{1}{2} \int_V d\tau \nabla \times J(\tau) + \chi_m \int_V d\tau \cdot R(\tau) \quad (62)
\]

where $V$ is the region occupied by the cylinder and $\chi_m$ is the magnetic susceptibility with respect to the background, defined as in (51). The first term in (62) vanishes as $\omega \rightarrow 0$ or $\sigma_1 \rightarrow 0$, while the second term becomes zero as $\omega \rightarrow \infty$ or $\mu_1 \rightarrow \mu$. A physical explanation and rigorous derivation of (62) starting from the equivalence principle is given in the Appendix.

The solution for the problem of a conducting and permeable infinite cylinder in a uniform alternating magnetic field is outlined in [13]. For a cylinder centered at $\rho = 0$, with its axis along $\xi$ and a longitudinal primary magnetic field as given by (5), the induced currents are

\[
\tilde{J}(\tau) = \hat{\phi}_0 H_0 e^{-k_1 J_1(k_1 \rho)} \frac{1}{k_1 J_0(k_1 a)} \quad (63)
\]

and the internal magnetic field is

\[
\tilde{H}(\tau) = \hat{\xi} H_0 e^{-k_1 J_1(k_1 \rho)} \frac{1}{k_1 J_0(k_1 a)} \quad (64)
\]

both given in terms of Bessel functions of the first kind with complex argument.

Inserting (63) and (64) into (62) and carrying out the integrations leads to the following expression for the magnetic dipole moment of the long cylinder under axial excitation

\[
\tilde{m}_l \approx -\hat{\tau} \pi a^2 \ell R_{\xi y A}^{\parallel} \quad (65)
\]

with

\[
R_{\xi y A}^{\parallel} = -\frac{k_1 a J_2(k_1 a) + 2 \chi_m J_1(k_1 a)}{k_1 a J_0(k_1 a)} \quad (66)
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\]

\(^6\)This methodology is similar to using an infinite-cylinder approximation for plane-wave scattering from finite-length dielectric cylinders employed in certain remote sensing applications [42]–[51]. independent of $\ell$. For $\omega \rightarrow \infty$, we have the normalization $R_{\xi y A}^{\parallel} \rightarrow 1$ and as $\omega \rightarrow 0$, we find $R_{\xi y A}^{\parallel} \rightarrow -\chi_m$, which coincides with the acicular low-frequency limit for the prolate spheroid (51). This suggests that the approximation introduced for long but finite-length cylinders can indeed be used to obtain an approximation of the magnetic dipole moment induced in long prolate spheroids under axial excitation by replacing $R_{3n0}^{\parallel}$ in (35) with $R_{\xi y A}^{\parallel}$ given by (66). The use of (35) rather than (65) pays tribute to the fact that the volumes of spheroid and cylinder of the same length $\ell$ are different and guarantees the exact dipole moment for the spheroid as $\omega \rightarrow \infty$.

V. NUMERICAL IMPLEMENTATION AND RESULTS

For the numerical results below, the infinite systems of (29) and (61) are truncated so that $m,n \leq 35$. The same truncation is used in the evaluation of the infinite series in (30). The problem is then solved by performing a singular-value decomposition of the resulting square system matrix [52] in order to guard ourselves against possible ill conditioning. For the associated Legendre functions with real argument and their first derivatives, we use the routines published in [29]. The auxiliary function (60) is evaluated using a Gauss-Legendre quadrature [53] with 50 points on the interval $0 < \eta < 1$.

It is advantageous to solve (29) for the product $A_{2n-1} R_{\xi y A}^{(2n-1)}$ rather than for $A_{2n-1}$. Note that knowing the product is sufficient for evaluating (30). This helps prevent overflows as $c_1$ grows larger by balancing the exponential growth of $T_{\xi y A}^{(2n-1)}$ with that of $R_{\xi y A}^{(2n-1)}$ [see also (41) and (42)]. The growing exponentials can be canceled explicitly by making use of exponential scaling when computing the spherical Bessel functions with complex argument [54] that are required in the expansion (39). A similar expansion is used for the derivative of the radial wavefunction with respect to $\xi$ where the derivative of the Bessel function is eliminated by using the appropriate recurrence relation [32]. We keep 35 terms of both infinite series in (39). The expansion coefficients $d_{2n-2}^{(2n-1)}$ are obtained with the help of a complex version of the corresponding real routine in [29] where we compute sequences of length 45. Note that the normalization of the coefficients requires another truncation of an infinite series [30]. The complex spheroidal eigenvalues required for the computation of the expansion coefficients, finally, are obtained using Hodge's method [55], where the characteristic values of a tridiagonal, complex symmetric matrix of size $40 \times 40$, ordered as a sequence with increasing real parts, are computed using a standard high-performance linear algebra routine [52].

One advantage of this approach is that no initial estimates of the eigenvalues are required [56], [57].

The numerical results are plotted versus induction number

\[
|k_1| a = \frac{2a}{d} |k_1| \quad (67)
\]

that, when the product $\omega \mu_2 \sigma_1$ in (9) is real, is given by

\[
|k_1| a = a \sqrt{\omega \mu_2 \sigma_1} \quad (68)
\]

\(^7\)Note that similar growing exponentials can and should be canceled also from (66).
Fig. 2. Magnetic polarizability factor $R_{\parallel}^{||}$ for conducting and permeable prolate spheroids under axial excitation as a function of the induction number $\nu_{1} / \mu$ for various elongations $\ell / 2a$ and fixed relative permeability $\mu_{1} / \mu = 1$, shown as solid curves where the evaluation based on the formulation with expanded spheroidal wavefunctions is possible. The curves are for elongations $\ell / 2a = 1, 1.5, 2, 4, 6, 8, 10$ and break off earlier the more elongated the spheroid. The dashed curves are obtained from an approximate solution to the boundary value problem obtained from the infinite system of (61) and are shown for the same elongations as the solid curves. The dash-dotted curves represent the corresponding $R_{\parallel}^{||}$ for long circular cylinders given in terms of Bessel functions.

(b) Imaginary part.

The induction number $|k_{1}|a$ is related to the skin depth $\delta_{\text{skin}}$ by

$$c_{1} = \frac{d\sqrt{1}}{2a} |k_{1}|a.$$  (69)

In Figs. 2–5, the computed real and imaginary parts of the polarizability factor $R_{\parallel}^{||}$ are shown separately, as a function of $|k_{1}|a$ and with the elongation $\ell / 2a$ as parameter. The different figures are for a relative permeability $\mu_{1} / \mu$ of 1, 10, 100, and 1000, respectively.

The solid curves in Figs. 2–5 were obtained from (29) and (30). These curves are truncated at that point where the numerical implementation of the expansion of the spheroidal wavefunctions into spherical harmonics is found to break down abruptly. The breakdown point occurs at a smaller induction number $|k_{1}|a$ for a larger elongation $\ell / 2a$ (the curves shown are for $\ell / 2a = 1, 1.5, 2, 4, 6, 8, 10$, respectively) but depends little on relative permeability $\mu_{1} / \mu$ (note the extended range of induction numbers in Fig. 5). In the spherical limit $\ell / 2a \rightarrow 1$, no such breakdown occurs and the results can be shown to be in agreement with the magnetic polarizability factor $R_{\parallel}^{\text{sph}}$ for conducting and permeable spheres of radius $a$, given by [10]–[14]

$$R_{\parallel}^{\text{sph}} = -\frac{(\mu_{1} + \mu)(1 - k_{3} \alpha \cot k_{3} \alpha)}{(\mu_{1} - \mu)(1 - k_{3} \alpha \cot k_{3} \alpha)}.$$  (71)

The dash-dotted curves in Figs. 2–5 were computed from (66) and can be considered as an approximation of $R_{\parallel}^{\text{sph}}$ as $\ell / 2a \rightarrow \infty$. They provide an important check of the behavior of the results as $\ell / 2a$ grows larger.

Since the low-frequency limit (49) extends to larger induction numbers as $\mu_{1} / \mu$ increases (Section II-D), the solid curves in Figs. 4 and 5 break off earlier than desirable. Thus, Figs. 2–5 were complemented with results from the approximate solution of the boundary value problem derived in Section III, shown as dashed curves. We can see that the low-frequency results (solid curves) are indeed extended to the high-frequency regime. The match in the overlap region is better for smaller $\ell / 2a$ and larger $\mu_{1} / \mu$. In contrast to results easily obtainable from the asymptotic formula (45), as $\ell / 2a$ increases in Figs. 4 and 5, the minimum of the imaginary part of $R_{\parallel}^{||}$ in Figs. 4(b) and 5(b) moves down into and toward the center of the trough representing the
Fig. 4. Similar to Fig. 2, but for a relative permeability of $\mu_1/\mu = 100$.

In general, independent of permeability, we expect (61) to yield accurate results when the skin depth is small compared to the radius of curvature everywhere along the surface of the spheroid, i.e.,

$$\delta_{\text{skin}} < \epsilon r_{\text{min}}$$  \hspace{1cm} (72)

where

$$r_{\text{min}} = a \frac{2\ell}{\ell^2}$$  \hspace{1cm} (73)

is the radius of curvature at the poles of the spheroid and, e.g., $\epsilon = 1/10$. Thus

$$|k_1|a > \frac{\sqrt{2}}{\epsilon} \frac{\ell}{2a}$$  \hspace{1cm} (74)

is required.

All curves plotted exhibit the general behavior expected for the physics described here. As frequency increases, we pass from the magnetostatic regime through the low-frequency scattering domain, characterized by magnetic diffusion effects, to the high-frequency limit that is determined by the induced surface eddy currents subjected to an extreme skin effect. The real part of $H_{\text{pro}}$ transits monotonically from the nonpositive low-frequency limit (49) to the positive high-frequency limit (37); the low-frequency limit decreases and approaches $\chi_{\text{pro}} = -(\mu_1/\mu - 1)$ as $\ell/2a$ increases. The imaginary part of $H_{\text{pro}}$ vanishes as $\omega \to 0$, $\infty$ and passes through a single minimum in between. The location of the minimum moves to smaller induction numbers as $\ell/2a$ increases and its absolute value increases, with a rate that is greater for larger permeability of the spheroid. The results show that the bandwidth of this minimum, for a permeable spheroid, decreases as the elongation $\ell/2a$ increases, with practical consequences pointed out in [58].

Since $H_{\text{pro}}$ for $\mu_1/\mu = 1$ (solid curves in Fig. 2) is found to be a relatively weak function of the elongation $\ell/2a$, in this case we can obtain a closed-form approximation of the induced magnetic dipole moment by replacing $H_{\text{pro}}$ in (35) by $R_{\text{sph}}$ given by (71). Note that the normalizing factor in (35) depends on $\ell/2a$ but is frequency-independent and given in terms of elementary functions.

In addition to observing the reasonable behavior of the previous results under variation of the parameters $\ell/2a$ and $\mu_1/\mu$, we validated the numerical implementation of our analytical solutions by comparing with results from MAS [9] (body-of-revolution code) and the approaches described in [8], [17]. The evaluation of the spheroidal wavefunctions and related quantities was verified against tabulated data in [29], [56], [59].
VI. BROADBAND RATIONAL FUNCTION APPROXIMATION

We have seen previously that it is possible to obtain accurate numerical results for $R_{\infty}$ from $\omega = 0$ up to a certain frequency $\omega_L$ and from some $\omega_H$ up to arbitrarily high frequencies. If $\omega_L < \omega < \omega_H$, the two data sets do not overlap and a gap for $\omega_L < \omega < \omega_H$ remains. To address this issue, we approximate an arbitrary polarizability factor by a rational function $R$ with $M$ simple poles and in the form of partial fractions, given by

$$R = 1 - \sum_{m=1}^{M} \frac{r_m}{1 - i\omega/\omega_m}$$  \hspace{1cm} (75)

where the $\omega_m$ are corner frequencies to be determined. The residues $r_m$ satisfy

$$\sum_{m=1}^{M} r_m = 1 - R_0$$  \hspace{1cm} (76)

where the low-frequency limit $R_0$ may be known in closed form from the corresponding magnetostatic problem, cf., (49). Note that automatically $R \rightarrow 1$ as $\omega \rightarrow \infty$ and the model is forced-stable if $\omega_m > 0$. Once available, $R$ lends itself to rapid evaluation in the frequency domain. Furthermore, simulations in the time domain can be carried out using recursive convolution [60].

A general physical justification of the model (75) is given by the singular expansion method (SEM) for the representation of magnetic polarizability tensors [1]. For the example of conducting and permeable spheres, (71) can be cast into the form of (75) with $M \rightarrow \infty$. For a sphere with $\mu_1/\mu = 1$ we have explicitly

$$r_m = \frac{6}{m^2 \pi^2}$$  \hspace{1cm} (77)

and

$$\omega_m = \frac{m^2 \pi^2}{2 \mu_1 \sigma_1}.$$  \hspace{1cm} (78)

For given discrete data $R(\omega)$ and a fixed $M$, we can find the $\omega_m$ and $r_m$ from the solution of the nonlinear least-squares problem corresponding to (75) with the linear constraint (76), employing standard numerical optimization procedures [61]. The elements of the Jacobian are found from (75) and given analytically by

$$\frac{\partial R}{\partial r_m} = \frac{1}{1 - i\omega/\omega_m}$$  \hspace{1cm} (79)

and

$$\frac{\partial R}{\partial \omega_m} = \frac{i\omega}{\omega_m^2} \frac{r_m}{(1 - i\omega/\omega_m)^2}.$$  \hspace{1cm} (80)

For simplicity, we suggest using a large model order, e.g., $M = 20$ and, as initial guess, distributing the $\omega_m$ uniformly on logarithmic scale [62]. This will lead to an accurate fit. If necessary for computational efficiency later, model-order reduction techniques can be employed subsequently.

For the example of a spheroid with $\ell/2a = 2$ and $\mu_1/\mu = 10$, the broadband approximation technique is demonstrated in Fig. 6. The data and the computed fit are plotted as solid and dash-dotted curves, respectively, overlaying each other. The low-frequency portion of the data was obtained from (29), (30). The high-frequency part was computed from (61) and here (74) with $\epsilon = 1/10$ holds. The $M = 20$ corner frequencies $\omega_m$ and residues $r_m$ are listed in Table I where

$$\Omega_m = \log_{10}(\omega_m^2 \mu_1 \sigma_1).$$  \hspace{1cm} (81)

Utilizing these data in conjunction with the simple model (75) provides rapid access to a nonspherical reference case with intermediate relative permeability when developing numerical codes that solve the problem of general objects. The largest absolute mismatches of the fit shown in Fig. 6 occur at the low-frequency end of the high-frequency approximation and are less than $8 \times 10^{-5}$ (0.8% of the high-frequency limit) for both real and imaginary part.
VII. CONCLUSION

By studying the approaches, solutions, and results described earlier, one can gain much insight into the nature of magnetic diffusion into and low-frequency scattering from nonspherical objects. Furthermore, the presented solution to a canonical problem has an immediate application in the validation of computer codes for the treatment of arbitrarily shaped objects. The UXO problem is only one possible motivation for this work. Other applications where nonspherical, conducting, and permeable “particles” play a role are easily conceivable.

We have seen that the exact solution of the boundary value problem based on spheroidal wavefunctions that are expanded in terms of spherical harmonics is directly usable numerically for small to intermediate induction numbers. A fundamental difficulty in this type of magnetoquasistatic problems, however, lies in the complex wavenumbers inside the object that vary over many orders of magnitude. For example, the truncation of the system matrix in Hodge’s method for determining the spheroidal eigenvalues (Section V) requires that the dimension of the matrix be much larger than \( |c_2| \) in (67). This limits the applicability of the solution as \( |c_2| \) grows exponentially. Furthermore, \( c_1 \) grows with equal real and imaginary parts. Thus, we engage a much less well-known domain of the spheroidal wavefunctions that may well be freighted with mathematical questions we have not touched upon in the course of our pragmatic development [59], [63]. Obtaining an approximate solution for high frequencies that avoids the necessity of evaluating the spheroidal wavefunctions altogether teaches the lesson that solving the Helmholtz equation with a reasonable degree of accuracy defies simplification beyond a certain point. The first such asymptotic solution, given in closed form and referring only to functions associated with solutions of Laplace’s equation, was found to be inaccurate for the frequencies of interest. However, based on a special thin-skin approximation we were able to construct an infinite system of equations that yields satisfactory results; in particular, this formulation gives a good broadband approximation in the case of large permeability. From the derivation it is apparent that the complexity here stems from the varying local geometry of the boundary. This complication disappears in the limit of a long spheroid and we gave an approximate closed-form solution that relies on Bessel functions rather than spheroidal wavefunctions. Finally, for practical purposes it is important to join low-frequency and high-frequency results together, in order to find the broadband response of the spheroid. We realized this by using a straightforward numerical fitting procedure based on rational functions with simple real poles. The partial fractions obtained can also readily be employed in time-domain analyses.

APPENDIX

We derive here equivalent surface-surface, volume-surface and volume-volume formulations for the magnetic dipole moment of conducting and permeable objects of arbitrary shape. Consider a homogeneous and isotropic body of a certain conductivity and permeability \( \mu_1 \) in a homogeneous, isotropic and nonconducting medium of permeability \( \mu_2 \). The body occupies the region \( \Omega_1 \) with surface \( S_1 \) and \( \hat{n} \) denotes the unit normal vector on \( S_1 \) pointing into the background medium, region \( \Omega_2 \). The total fields everywhere are assumed to be known.

According to the equivalence principle [64], the secondary fields in \( \Omega_2 \) can be thought of being due to the fictitious electric surface current \( \hat{\mathbf{h}} \times \mathbf{H} \), and the fictitious magnetic surface current \( -\mathbf{h} \times \mathbf{E} \), both on \( S_1 \) across which these quantities are continuous. Electric loop currents and magnetic currents with nonzero divergence, the latter leading to magnetic charge accumulations, both contribute to the dipole moment [33] of the body, expressed as

\[
\mathbf{m} = \frac{1}{2} \int_{S_1} dS \mathbf{\tau} \times (\mathbf{h} \times \mathbf{H}_2) + \frac{1}{\omega \mu_2} \int_{S_1} dS \nabla_8 \cdot (-\mathbf{h} \times \mathbf{E}_2). \tag{82}
\]

For relations involving the surface Nabla operator \( \nabla_8 \), see Appendix 2 in [12]. Noting that

\[
i \omega \mu_2 \hat{\mathbf{h}} \cdot \mathbf{H}_2 = \mathbf{h} \cdot (\nabla \times \mathbf{E}_2) = \mathbf{h} \cdot (\nabla_8 \times \mathbf{E}_2) \tag{83}
\]

\[
= \nabla_8 \cdot (\mathbf{E}_2 \times \hat{\mathbf{n}}) + \mathbf{E}_2 \cdot (\nabla_8 \times \hat{\mathbf{n}}) \tag{84}
\]

\[
= \nabla_8 \cdot (-\mathbf{h} \times \mathbf{E}_2) \tag{85}
\]

we find from (82)

\[
\mathbf{m} = \frac{1}{2} \int_{S_1} dS \mathbf{\tau} \times (\mathbf{h} \times \mathbf{H}_2) + \int_{S_1} dS \nabla_8 \cdot \mathbf{H}_2. \tag{87}
\]

Considering the Maxwell equations \( \nabla \times \mathbf{H} = \mathbf{J} \) and \( \nabla \cdot \mathbf{D} = -\mathbf{E} / \varepsilon \mu / \mu \), we can alternatively view the dipole moment as being produced by the physical eddy currents circulating in the body and fictitious magnetic surface charges at the surface of discontinuity of \( \mu \), i.e., \( S_1 \). In this picture, we write

\[
\mathbf{m} = \frac{1}{2} \int_{\Omega_1} dV \mathbf{\tau} \times \mathbf{J}_1 + \int_{S_1} dS \nabla_8 \cdot (\mathbf{H}_2 - \mathbf{H}_1). \tag{88}
\]

In the following, we show that (88) and (87) are indeed equivalent. Using

\[
\nabla (\mathbf{\tau} \cdot \mathbf{H}_1) = (\mathbf{\tau} \cdot \nabla) \mathbf{H}_1 + (\mathbf{H}_1 \cdot \nabla) \mathbf{\tau} + \mathbf{\tau} \times (\nabla \times \mathbf{H}_1) + \mathbf{H}_1 \times (\nabla \times \mathbf{\tau}) \tag{89}
\]

\[
= \nabla \cdot (\mathbf{\tau} \mathbf{H}_1) - 2 \mathbf{H}_1 + \mathbf{\tau} \times \mathbf{J}_1 \tag{90}
\]

we find from (82)

\[
\mathbf{m} = \frac{1}{2} \int_{S_1} dS \nabla_8 \cdot \mathbf{H}_1 - \frac{1}{2} \int_{S_1} dS \nabla_8 \cdot \mathbf{H}_1 \tag{93}
\]

which is seen to be same as (87).

Through (87), one obtains \( \mathbf{m} \) from surface integration while (88) is the sum of a volume and a surface integral. We can rewrite (88) as

\[
\mathbf{m} = \frac{1}{2} \int_{\Omega_1} dV \mathbf{\tau} \times \mathbf{J}_1 + \frac{\mu_1 - \mu_2}{\mu_2} \int_{S_1} dS \nabla_8 \cdot \mathbf{H}_1 \tag{94}
\]

\[
= \frac{1}{2} \int_{\Omega_1} dV \mathbf{\tau} \times \mathbf{J}_1 + \frac{\mu_1 - \mu_2}{\mu_2} \int_{\Omega_1} dV \mathbf{\nabla} \cdot (\mathbf{H}_1 \mathbf{\tau}) \tag{95}
\]
which gives the pure volume integral formulation

$$\mathbf{m} = \frac{1}{2} \int_{V_1} \mathbf{d}^2 \mathbf{r} \times \mathbf{J}_1 + \frac{\rho_1 - \rho_2}{\rho_2} \int_{V_1} \mathbf{d}^2 \mathbf{G}_1.$$  (96)

The second term in (96), adding to the dipole moment of the physical eddy currents, can be viewed as arising from the presence of volumetric magnetic \"contrast sources.\" 

REFERENCES


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